PHYS 4410

Quantum Mechanics 2

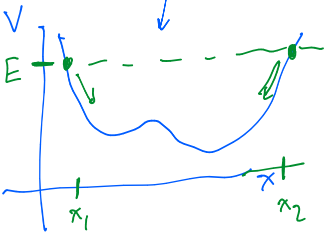
Spring 2023

Lecture 33

The WKB approximation

April 14

$\boxed{1}$ Semiclassical limit of 1d Schrödinger eqn:



Particle of mass m, energy E.

$V(x_1) = V(x_2) = E$.

Roll back and forth.

Period of oscillations:

$$T = T_{\rightarrow} + T_{\leftarrow} = 2T_{\rightarrow} = \int_{x_1}^{x_2} \overbrace{\frac{dx}{v(x)}}^{dt} = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \underbrace{\frac{dx}{\sqrt{E - V(x)}}}_{}.$$

$T_{\rightarrow}$: move right

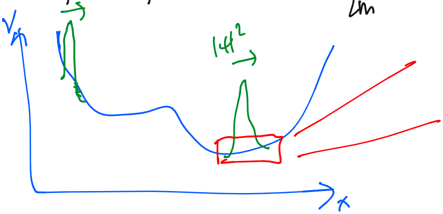
$T_{\leftarrow}$: move left

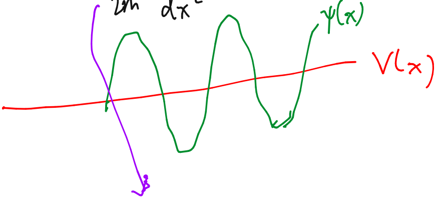
$E = \frac{1}{2} m v(x)^2 + V(x)$

integral converges if $V'(x_1), V'(x_2) \neq 0$.

$\boxed{2}$ Time-independent Schrödinger equation:

$H\psi = E\psi \longrightarrow H = \frac{p^2}{2m} + V(x)$: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$.

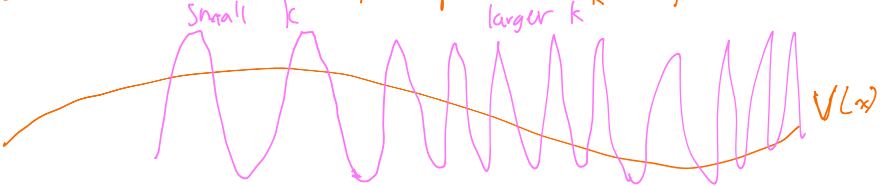




If $V(x) = V_0$ (const.): $\psi = e^{ikx}$: $\frac{\hbar^2}{2m}k^2 + V_0 = E$.

or $k = \frac{\sqrt{2m(E-\cancel{V_0})}}{\hbar}$ $V(x)$

Idea: if $V(x)$ varies slowly compared to $1/k$: $\psi(x) =$ "$e^{ik(x)\cdot x}$"



$\boxed{3}$ Formalize: WKB approximation. Let $\psi(x) = A(x)\, e^{iS(x)/\hbar}$

Take "$\hbar \to 0$" (semiclassical):

$$\underbrace{\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(Ae^{iS/\hbar}\right)} = [E - V(x)]\cdot Ae^{iS/\hbar}$$

If $\hbar \to 0$ limit, LHS $\to 0$ unless:

$$\frac{d}{dx}\left[\cancel{\frac{dA}{dx}e^{iS/\hbar}} + \frac{i}{\hbar}\frac{dS}{dx}Ae^{iS/\hbar}\right] \simeq \left(\frac{i}{\hbar}\frac{dS}{dx}\right)^2 Ae^{iS/\hbar} + \frac{1}{\hbar}[\cdots]$$

$$-\frac{\hbar^2}{2m}\left(-\frac{1}{\hbar^2}\left(\frac{dS}{dx}\right)^2 + \frac{1}{\hbar}[\cdots] + \cdots\right)\cancel{Ae^{iS/\hbar}} = [E - V(x)]\,\cancel{Ae^{iS/\hbar}}$$

Now take $\hbar \to 0$: $\frac{1}{2m}\left(\frac{dS}{dx}\right)^2 = E - V(x)$

Define $p(x)^2 = 2m(E - V(x))$: $\frac{dS}{dx} = \pm\sqrt{p(x)^2} = \pm p(x)$

if $p \simeq$ const.

So: $S(x) = \int_0^x dx'\, p(x') \longrightarrow p(0)\cdot x$, $\psi = A\cdot \underbrace{e^{ip(0)x/\hbar}}_{\text{plane wave}}$

$\boxed{4}$ Also, WKB allows corrections: $\psi(x) = e^{i\left(S_0/\hbar + S_1 + S_2\hbar + \cdots\right)}$

$$\frac{d^2\psi}{dx^2} = \left[-\frac{1}{\hbar^2}\left(\frac{dS_0}{dx}\right)^2 + \frac{i}{\hbar}\left[-2\frac{dS_0}{dx}\frac{dS_1}{dx} + i\frac{d^2S_0}{dx^2}\right] + \cancel{\hbar^0[\cdots]}\right]\psi(x)$$

$\frac{dS_0}{dx} = p(x)$: $\quad -\frac{\hbar^2}{2m}\left[-\cancel{\frac{1}{\hbar^2}p^2} + \frac{1}{\hbar}\underbrace{\left[-2p\frac{dS_1}{dx} + i\frac{dp}{dx}\right]}_{=0\text{, so expansion is as accurate.}} + \cdots\right] = (\cancel{E-V})$

So: $\frac{dS_1}{dx} = \frac{i}{2}\cdot\frac{1}{p}\frac{dp}{dx}$.

$$S_1(x) = \frac{i}{2}\int \cancel{dx}\,\frac{1}{p}\frac{dp}{\cancel{dx}} = \frac{i}{2}\int\frac{dp}{p} = \frac{i}{2}\log p \quad (\text{or } \cancel{\ln p})$$

$$\hookrightarrow \psi(x) \approx e^{iS_0/\hbar + iS_1} = e^{iS_0/\hbar - \frac{1}{2}\log p} = \frac{e^{i/\hbar\int dx\, p}}{\sqrt{p(x)}}$$

Approximation accurate if:

NOT near turning points (x_1, x_2): $\quad p(x) \to 0$.

$\boxed{5}$ WKB be a good approx if:

"$S_0/\hbar \gg S_1$" $\longrightarrow$ $\left|\frac{1}{\hbar}\frac{dS_0}{dx}\right| \gg \left|\frac{dS_1}{dx}\right|$

L = length scale over which $p(x)$

$$\frac{p(x)}{\hbar} \gg \left|\frac{d}{dx}\frac{1}{2}\log p\right| = \frac{1}{2}\underbrace{\frac{1}{p}\frac{dp}{dx}}_{1/L}$$

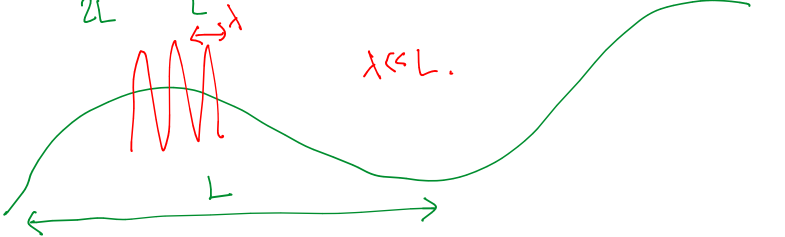
$\frac{1}{p} \sim \lambda$

wavelength of plane wave

$$p(x) \gg \frac{\hbar}{2L} \sim \frac{\hbar}{L}$$

$V(x)$

$\psi(x)$



$\lambda \ll L$.