PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 34

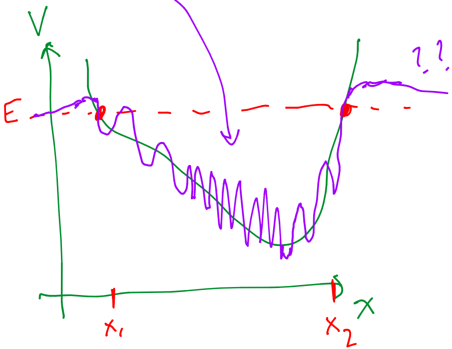
The Bohr-Sommerfeld approximation

April 19

1 Review: WKB approx:

$$\psi(x) = \frac{e^{i\int dx\, p(x)/\hbar}}{\sqrt{p(x)}}, \quad p(x) = \frac{\sqrt{2m(E-V(x))}}{\hbar}, \qquad \text{valid: } \frac{p}{\hbar} \gg \left|\frac{1}{p}\frac{dp}{dx}\right|$$

(wavelength) $\lambda^{-1} \gg 1/L$ (potential varying scale)



$V(x_1) = E = V(x_2)$

turning points

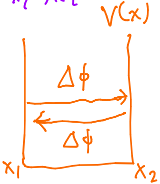
In QM:

- for bound state:

$$\underbrace{\Delta\phi_{\rightarrow}}_{\substack{\text{phase}\\ x_1 \to x_2}} + \underbrace{\Delta\phi_{\leftarrow}}_{\substack{\text{phase}\\ x_2 \to x_1}} = 2\pi n$$

integer

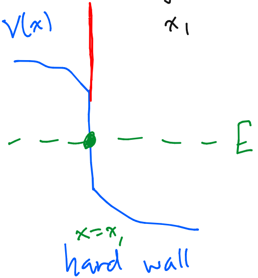
E.g.

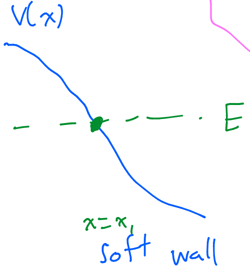


$$2\pi n = 2\Delta\phi$$
$$= 2\cdot k(x_2 - x_1)$$

$\boxed{2}$ $\quad \Delta\phi_{\to} = \Delta\phi_{\leftarrow}$

$$2\Delta\phi_{\to} = 2\int_{x_1}^{x_2} dx\, \frac{p(x)}{\hbar} + \boxed{2\,\text{boundary terms}} = 2\pi n$$





Analysis of soft: add phase $-\pi/2$

net bdy effect:

2 hard walls: bdy term $= -2\pi$

2 soft walls: bdy $= -\pi$

If $V(x) = \infty$ for $x < x_1$: $\psi(x_1) = 0$.

$$\psi(x) \simeq \sin\left[\int_{x_1}^{x} dx'\, \frac{p(x')}{\hbar}\right] \frac{1}{\sqrt{p}}$$

$$= \frac{1}{\sqrt{p}}\left[\frac{1}{2i}\underbrace{e^{i\int p/\hbar}}_{\text{right}} - \frac{1}{2i}\underbrace{e^{-i\int p/\hbar}}_{\text{left}}\right]$$

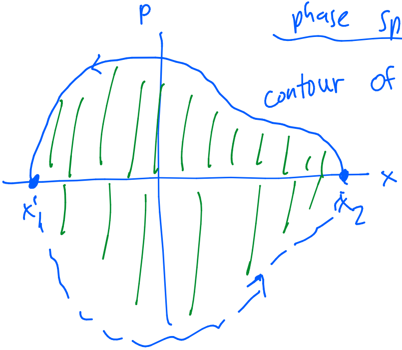
bdy term: add phase $-\pi$

3 Conclusion:

$$2\int_{x_1}^{x_2} dx \boxed{\begin{matrix} p(x) \\ \shortparallel \\ \sqrt{2m(E-V(x))} \end{matrix}} = 2n\pi\hbar + 2\begin{cases} \pi & \text{2 hard walls} \\ 3\pi/4 & \text{1 hard / 1 soft} \\ \pi/2 & \text{2 soft} \end{cases}$$

$n = 0, 1, 2, 3, \ldots$

Bohr-Sommerfeld approx: allowed energy levels E_n
solve eqn for $n = 0, 1, 2, \ldots$



phase space

contour of $H(x,p) = E$.

$$H(x,p) = \frac{p^2}{2m} + V(x)$$

B-S approx:

$$\text{Area(phase space)} = 2\pi\hbar \begin{cases} n+1 \\ n+3/4 \\ n+1/2 \end{cases}.$$

$\boxed{4}$ $H = \frac{p^2}{2m} + F|x|$ (HW7)

B-S: • $\int_{x_1}^{x_2} dx\, p(x) = \pi\hbar\left(n + \frac{1}{2}\right)$

$E = F|x|$; $x_1 = -E/F$

$x_2 = +E/F$

• If E fixed: find x_1 & x_2: $E = V(x_1) = V(x_2)$

• $\pi\hbar\left(n+\frac{1}{2}\right) = \int_{-E/F}^{E/F} dx \sqrt{2m(E - F|x|)} = 2\int_0^{E/F} dx \sqrt{2m(E-Fx)}$

$2 \cdot \frac{1}{F}\sqrt{2m}\,\frac{2}{3}(E-Fx)^{3/2}\Big|_0^{E/F} = \frac{4}{3F}\sqrt{2m}\; E^{3/2}$

$E_n^{3/2} = (n+1/2)\,\pi\hbar\,\frac{3F}{\sqrt{32m}}$

$E_n = \left[\left(\cancel{n}+\frac{1}{2}\right)^2 \frac{\hbar^2 F^2}{m} \cdot \frac{9\pi^2}{32}\right]^{1/3}$

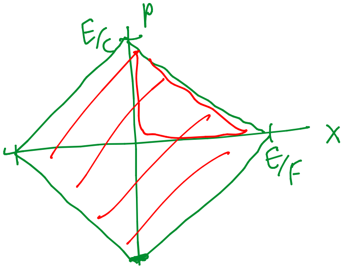
$\to E_0 \approx \left(\frac{\hbar^2 F^2}{m}\right)^{1/3} \cdot 0.89$

HW7 (variational):

$E_0 \leq \left(\frac{F^2\hbar^2}{m}\right)^{1/3} \cdot \underbrace{\left(\frac{81}{128}\right)^{1/3}}_{0.86}$

5 B-S approx can be applied if $H(p,x)$

Example: $H = c|p| + F|x|$ (ultrarel. / massless particle: meson binding).



$$\text{Area} = 2\pi\hbar(n+1/2)$$
$$= 4 \times \frac{1}{2}\frac{E}{F}\frac{E}{c}$$

Solve:

$$E_n \approx \sqrt{\hbar c F_\pi (n+1/2)}$$