PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 35

Tunneling

April 21

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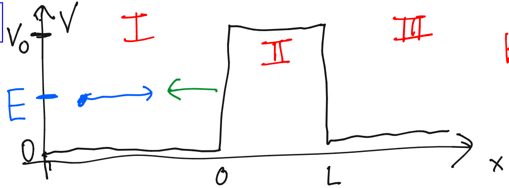

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

$$H\psi = E\psi$$

classically: bounce off since in II: $E < V(x)$

quantumly: solve for $\psi(x)$:

$$\psi(x) = \begin{cases} e^{ikx} + A e^{-ikx} & \text{I} \\ \cancel{Ce^{\kappa x}} \; De^{-2\kappa L} + De^{-\kappa x} & \text{II} \\ Be^{ikx} & \text{III} \end{cases}$$

$\kappa L \gg 1$

At $x = 0$ & L:
continuity of ψ and $\frac{d\psi}{dx}$

$x = L$:

ψ: $Be^{ikL} = Ce^{\kappa L} + De^{-\kappa L}$

& ψ': $B \sim Ce^{\kappa L} \sim De^{-\kappa L}$

$$De^{-2\kappa L} \sim C$$

& III

I: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

II: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = -(V_0 - E)\psi$

$$\kappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

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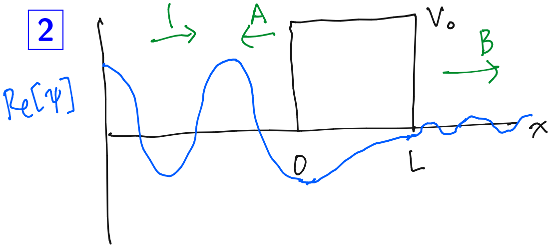

By matching:

$$B \sim D e^{-\kappa L}$$

$$D \sim A \sim 1$$

quantum tunneling.

Probability to tunnel (appear on right):

$$(T) = P_{tunnel} = |B|^2$$

$$P_{tunnel} \sim \# \cdot e^{-2\kappa L}$$

$$\sim \# e^{-2 \frac{\sqrt{2m(V_0 - E)}}{\hbar} L}$$

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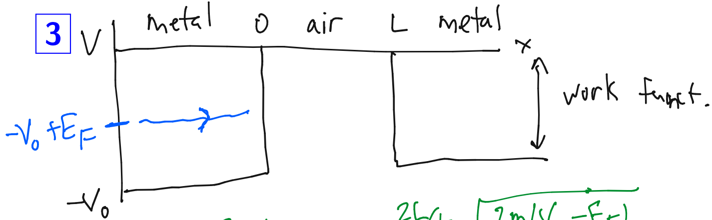

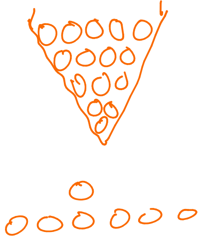

$$P_{tunnel} = e^{-2\kappa L} = e^{-\frac{2L}{\hbar}\sqrt{2m(V_0-E_F)}}$$

Ex numbers: $m \sim 10^{-31}$ kg ; $E_F = \frac{V_0}{2} = 1.5\,eV$

$= 3\times10^{-19}\,J$

$[\hbar \sim 10^{-34}\,J\cdot s]$

For what L will $P_{tunnel} \sim 10^{-3}$?

$$10^{-3} \sim e^{-10}$$

$$10 \sim \frac{2L}{\hbar}\sqrt{2mE_F} \sim \sqrt{8}\,L\,\frac{\sqrt{10^{-31}\cdot 3\times10^{-19}}}{10^{-34}}$$; $L \sim 2\,nm$ $(2\times10^{-9}\,m)$

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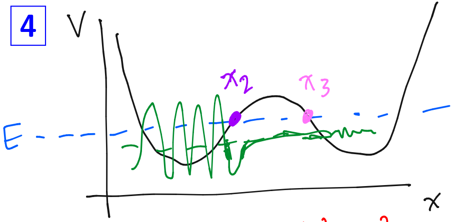

What fraction of $\psi(x)$ leak into right?

Follow WKB! $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$ $\quad V(x) > E$

$$\underbrace{\frac{2m}{\hbar^2}\left[V(x) - E\right]}_{\kappa(x)^2}\psi(x) = \frac{d^2\psi}{dx^2}$$

Approximate?

Guess: $\psi(x) \approx e^{-\int_{x_2}^{x} dx' \, \kappa(x')}$

Similar to before: accurate if

$$\left|\frac{1}{\kappa}\frac{d\kappa}{dx}\right| \ll \kappa$$

$$P_{tunnel} \approx e^{-2\Gamma}$$

$$\Gamma = \frac{1}{\hbar}\int_{x_2}^{x_3} dx \sqrt{2m(V(x)-E)}$$

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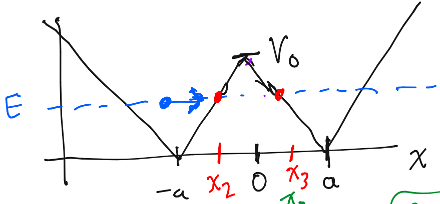

$$V(x) = \frac{V_0}{a}\Big|\, a - |x| \,\Big|$$

$$-x_2 = x_3$$

$$P_{\text{tunnel}} = e^{-2\Gamma}, \quad \Gamma = \int_{x_2}^{x_3} dx \, \frac{\sqrt{2m(V(x)-E)}}{\hbar}$$

$$V(x) = E = \frac{V_0}{a}(a - x_3). \qquad \frac{Ea}{V_0} = a - x_3. \qquad x_3 = a\left(1 - \frac{E}{V_0}\right)$$

$$\Gamma = 2\int_0^{x_3} dx \, \frac{\sqrt{2m\, V_0/a\left[(a-x) - (a-x_3)\right]}}{\hbar} = 2\int_0^{x_3} dx \, \frac{\sqrt{2mV_0}}{\hbar\sqrt{a}}\sqrt{x_3 - x}$$

$$= \frac{2}{\hbar}\sqrt{\frac{2mV_0}{a}} \cdot \frac{2}{3} x_3^{3/2} = \frac{4a}{3\hbar}\sqrt{2mV_0}\left(1 - \frac{E}{V_0}\right)^{3/2}.$$