PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 37

The adiabatic theorem

April 26

$\boxed{1}$ Adiabatic quantum computer:

$$H(t) = (1 - t/\tau) H_0 + t/\tau \, H_p \qquad (0 \le t \le \tau)$$

problem Hamiltonian:

$$H_0 = -\sum_{i=1}^{N} \sigma_i^x$$

$$H_p = \sum_{i,j=1}^{N} J_{ij} \sigma_i^z \sigma_j^z$$

ground state of H_0:

$$|\psi_0\rangle = |\uparrow_x \uparrow_x \cdots \uparrow_x\rangle$$

$$|\uparrow_x\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}.$$ ← z-basis

- ground state of H_p encode sol'n to hard combinatorial...
- suppose g.s. unique $|\psi_p\rangle$.

Adiabatic Thm

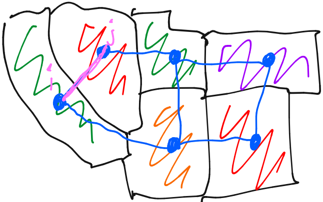
Claim: Suppose $|\psi(t=0)\rangle = |\psi_0\rangle$.

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar} H(t)|\psi(t)\rangle.$$

how? lec 38.

For τ sufficiently large, $|\psi(t=\tau)\rangle \approx |\psi_p\rangle$.

$\boxed{2}$ Graph [map] coloring.



Math problem:

Two vertices connected by edge assigned diff colors.

Is this possible given q colors?

$\alpha = \{1, \dots q\}$

Encode in H_p?

$$H_p = \sum_{\substack{\text{edges} \\ ij}} U_{ij} + \sum_{\substack{\text{vertex} \\ i}} C_i$$

$$U_{ij} = \begin{cases} 1 & i,j \text{ same} \\ 0 & \text{not} \end{cases}$$

$$C_i = \begin{cases} >0 & i \text{ has not } \cancel{\text{unique}} \text{ color} \\ 0 & \end{cases}$$

Let $\sigma^z_{i\alpha} = \begin{cases} 1 & \text{if } i \text{ color } \alpha \\ -1 & \text{else} \end{cases}$

$$C_i = \left(q - 2 + \sum_{\alpha=1}^{q} \sigma^z_{i\alpha}\right)^2.$$

$$U_{ij} = \frac{1}{4} \sum_{\alpha=1}^{q} (1 + \sigma^z_{i\alpha})(1 + \sigma^z_{j\alpha})$$

$\boxed{3}$ Why do we stay in ground state (if τ large):

Consider $H(t) = H(\underline{\lambda_\alpha(t)})$

e.g. $H(t) = A(t)\sigma^x + B(t)\sigma^z$: $\{\lambda_\alpha\} = \{A, B\}$.

Let $H(\lambda)|\psi_n(\lambda)\rangle = E_n(\lambda)|\psi_n(\lambda)\rangle$.

We can always write:

$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}H(t)|\psi(t)\rangle$:

$$|\psi(t)\rangle = \sum_n a_n(t)\, e^{i\zeta_n(t)}\, |\psi_n(\lambda_\alpha(t))\rangle$$

$$\zeta_n(t) = -\frac{1}{\hbar}\int_0^t ds\, E_n(\lambda_\alpha(s)).$$

$\boxed{4}$ Plug into Schrödinger:

$$i\frac{d\xi_n}{dt} = -i E_n/\hbar$$

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

$$i\hbar\left[\sum_n \frac{da_n}{dt} e^{i\xi_n}|\psi_n\rangle + \cancel{i\frac{d\xi_n}{dt} a_n e^{i\xi_n}|\psi_n\rangle} + a_n e^{i\xi_n}\sum_\alpha \frac{d\lambda_\alpha}{dt}\frac{\partial}{\partial\lambda_\alpha}|\psi_n\rangle\right]$$

$$= \cancel{\sum_n E_n(\lambda)\, a_n e^{i\xi_n}|\psi_n\rangle}$$

$$\langle\psi_n|\left[\ \cdots\ \right] = 0 :$$

$$\frac{da_n}{dt} e^{i\xi_n} + \underbrace{\sum_m a_m e^{i\xi_m}}_{(m=n)\ +\ (m\neq n)} \sum_\alpha \frac{d\lambda_\alpha}{dt}\langle\psi_n|\frac{\partial}{\partial\lambda_\alpha}|\psi_m\rangle = 0 .$$

small if τ large

$\boxed{5}$ Claim: if $m \neq n$: "$\langle \psi_m | \frac{\partial}{\partial \lambda_\alpha} | \psi_n \rangle$ small"

$$\frac{\partial}{\partial \lambda_\alpha}\left[H|\psi_n\rangle = E_n |\psi_n\rangle \right]$$

$$\langle \psi_m | \left[\frac{\partial H}{\partial \lambda_\alpha} |\psi_n\rangle + H \frac{\partial |\psi_n\rangle}{\partial \lambda_\alpha} = \frac{\partial E_n}{\partial \lambda_\alpha} |\psi_n\rangle + E_n \frac{\partial |\psi_n\rangle}{\partial \lambda_\alpha} \right]$$

$$\langle \psi_m | H = E_m \langle \psi_m |$$

$$\langle \psi_m | \frac{\partial H}{\partial \lambda_\alpha} |\psi_n\rangle + E_m \langle \psi_m | \frac{\partial |\psi_n\rangle}{\partial \lambda_\alpha} = \cancel{\frac{\partial E_n}{\partial \lambda_\alpha} \langle \psi_m | \psi_n \rangle}^{\,0} + E_n \langle \psi_m | \frac{\partial |\psi_n\rangle}{\partial \lambda_\alpha}$$

$$\langle \psi_m | \frac{\partial H}{\partial \lambda_\alpha} |\psi_n\rangle = (E_n - E_m) \langle \psi_m | \frac{\partial}{\partial \lambda_\alpha} |\psi_n\rangle .$$

$\boxed{6}$

$$\frac{da_n}{dt} e^{i\zeta_n} = -\sum_\alpha \frac{d\lambda_\alpha}{dt}\left[a_n e^{i\zeta_n}\langle\psi_n|\frac{\partial}{\partial\lambda_\alpha}|\psi_n\rangle + \cancel{\sum_{m\neq n} a_m e^{i\zeta_m}\langle\psi_m|\frac{\partial}{\partial\lambda}|\psi_n\rangle}\right]$$

if $E_n \neq E_m$:

$$\sum_\alpha \frac{d\lambda_\alpha}{dt}\langle\psi_m|\frac{\partial}{\partial\lambda_\alpha}|\psi_n\rangle = \sum_\alpha \frac{\frac{d\lambda_\alpha}{dt}}{E_n-E_m}\langle\psi_m|\frac{\partial H}{\partial\lambda_\alpha}|\psi_n\rangle = \frac{1}{E_n-E_m}\langle\psi_m|\frac{dH}{dt}|\psi_n\rangle$$

$$H = H(t/\tau)$$

$$\cancel{\frac{1}{E_n-E_m}\cdot\frac{1}{\tau}\langle\psi_m|H'(t/\tau)|\psi_n\rangle}.$$

Adiabatic Thm:
if $\tau\to\infty$

$$\frac{da_n}{dt} = -a_n\sum_\alpha \frac{d\lambda_\alpha}{dt}\underbrace{\langle\psi_n|\frac{\partial}{\partial\lambda_\alpha}|\psi_n\rangle}_{\text{Show: } = -i\underbrace{A_\alpha}_{\text{real}}}$$

$$a_n(\tau) = \exp\left[i\int_0^\tau dt\sum_\alpha\frac{d\lambda_\alpha}{dt}A_\alpha\right] = \exp\left[i\int d\lambda_\alpha A_\alpha\right].$$