PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 38

Landau-Zener transitions

April 28

$\boxed{1}$ Review: adiabatic thm.

$$H(\lambda^\alpha)\,|\psi_n(\lambda^\alpha)\rangle = E_n(\lambda)\,|\psi_n(\lambda)\rangle$$

If $\lambda^\alpha(t)$ time-dependent... $\frac{d\lambda^\alpha}{dt} \sim \frac{1}{\tau}$ small; then

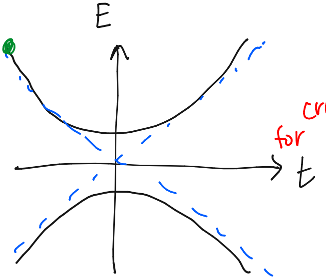
if $|\psi(0)\rangle = |\psi_n(\lambda=0)\rangle$, then $\underbrace{|\psi(t)\rangle \approx |\psi_n(\lambda(t))\rangle}_{\text{adiabatic: stay in eigenstate}}$

How large does τ need to be?

Claim: adiabatic thm hold if

$$\text{"}\quad 1 \ll \frac{|\langle\psi_m(t)|\,H(t+\tau)\,|\psi_n(t)\rangle|^2 \quad (m\neq n)}{\hbar\left|\frac{d}{dt}(E_m - E_n)\right|}\quad\text{"}$$

$\boxed{2}$ Adiabatic approx first fail at avoided crossing:



$$H = \begin{pmatrix} -\alpha t/\tau & \beta \\ \beta & \alpha t/\tau \end{pmatrix} \rightarrow E = \pm\sqrt{(\alpha t/\tau)^2 + \beta^2}$$

crucial for ad. thm.

If adiabatic approx holds:

$$|\psi(-\infty)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi(+\infty)\rangle \simeq \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\theta}$$

Note: if $\beta = 0$, then $|\psi(+\infty)\rangle \simeq e^{i\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

If $\underbrace{\beta \neq 0}_{\text{small; use time-dep PT?}}$: $|\psi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$.

$$i\hbar \frac{da}{dt} = -\frac{\alpha t}{\tau} a + \beta b$$

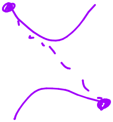
$$i\hbar \frac{db}{dt} = \frac{\alpha t}{\tau} b + \beta a$$

At leading order ($\beta = 0$):

$$i\hbar \frac{da}{dt} = -\underbrace{\alpha t/\tau}\, a \quad : \quad a(t) = e^{+i\alpha t^2/2\hbar\tau}$$

3 Now at first order:

$$\frac{db}{dt} = -\frac{i}{\hbar}\beta\, a(t) - \frac{i\alpha t}{\tau\hbar}\, b \,; \qquad \underbrace{\frac{db}{dt} + \frac{i\alpha t}{\tau\hbar}\, b = -\frac{i\beta}{\hbar}\, e^{i\alpha t^2/2\hbar\tau}}$$



Landau-Zener transition.

$$\frac{d}{dt}\left(b\, e^{i\alpha t^2/2\hbar\tau} \right) = -\frac{i\beta}{\hbar}\, e^{i\alpha t^2/\hbar\tau}$$

$$b(t) \approx -\frac{i\beta}{\hbar}\, e^{-i\alpha t^2/2\hbar\tau} \underbrace{\int_{-\infty}^{t} ds\; e^{i\alpha s^2/\hbar\tau}}$$

$$\sqrt{\frac{\alpha}{\hbar\tau}}\, s = z \, :$$

$$\sqrt{\frac{\hbar\tau}{\alpha}} \left[\int_{-\infty}^{\cancel{t\sqrt{\hbar\tau/\alpha}}\; +\infty} dz\; e^{iz^2} \right] \to \text{const.} = C$$

As $t \to \infty$...

$\downarrow$

$$b(+\infty) \approx -\frac{i\beta}{\hbar}\, e^{-i\alpha t^2/2\hbar\tau}\, C \sqrt{\frac{\hbar\tau}{\alpha}}\,.$$

Prob. of trans:

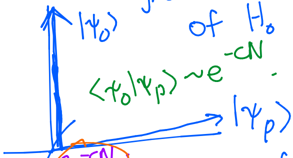
$$|b|^2 = \boxed{\frac{\beta^2 \tau}{\hbar \alpha}}\, |C|^2$$

PT fails when $\beta^2 \gtrsim \hbar\alpha/\tau$

≈ 1 if ad. thm holds. (PT fails)

$\boxed{4}$ Example: adiabatic quantum computer:

$$H = (1 - t/\tau) H_0 + t/\tau\, H_p$$



ground state of H_0

$\langle \psi_0 | \psi_p \rangle \sim e^{-cN}$.

g.s. of H_p

approx:

$$H = \begin{pmatrix} t/\tau & e^{-cN} \\ e^{-cN} & 1 - t/\tau \end{pmatrix} A = \cancel{\frac{A}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + \frac{A}{2}\begin{pmatrix} -1 + t/\tau & 2e^{-cN} \\ 2e^{-cN} & 1 - t/\tau \end{pmatrix}$$

How large does τ need to be for $|\psi(\tau)\rangle \approx |\psi_p\rangle \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ if $|\psi(0)\rangle = |\psi_0\rangle \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$1 \ll \frac{(2e^{-cN} A)^2}{\hbar \cdot \frac{d}{dt}\left(2t\,/\,\frac{A}{\tau}\right)} \sim \frac{A}{\hbar} e^{-2cN} \tau$$

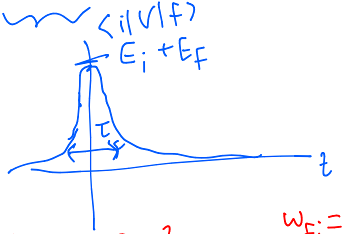
$$\tau \sim e^{2cN} \cdot \frac{\hbar}{A}$$

adiabatic algorithm:
$\tau \gtrsim e^{2cN}$.

Classical algorithm also runs in $t \sim e^{N}$.

5 Consider: $H(t) = H_0 + V(t/\tau)$

If τ small enough, and
$|\psi(-\infty)\rangle = |i\rangle$, can
$|\psi(+\infty)\rangle \approx |i\rangle$?



Again use time-dep PT: $|i\rangle \to |f\rangle$?

$$\omega_{fi} = \frac{E_f - E_i}{\hbar}$$

$$c_f^{(1)} = -\frac{i}{\hbar}\int_{-\infty}^{t} ds\; e^{i\omega_{fi}s}\langle f|V(s/\tau)|i\rangle \qquad (z = s/\tau)$$

$$= -\frac{i\tau}{\hbar}\int_{-\infty}^{t/\tau} dz\; e^{i\omega_{fi}\tau z}\langle f|V(z)|i\rangle .$$

As $\tau \to 0$:

$$P_{i\to f} \lesssim \left|\frac{\tau}{\hbar}\langle f|V|i\rangle\right|^2$$

small if $\tau < \dfrac{\hbar}{|\langle f|V|i\rangle|}$.

As $t \to \infty$:

$$c_f^{(1)} = -\frac{i\tau}{\hbar}\left[\int_{-\infty}^{\infty} dz\; e^{i\omega_{fi}\tau z}\langle f|V(z)|i\rangle\right]$$

$\sim \langle f|V|i\rangle$ (at $t=0$)

$\boxed{6}$ Example: $H(t) = \underbrace{A\vec{S}\cdot\vec{I}}$ $+ \hbar B S_z \underbrace{e^{-|t|/\tau}}$.

hyperfine; $\vec{S}, \vec{I}$ are spin-1/2

If $|\psi(-\infty)\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$, what's $\left| \langle\psi(\infty)| \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \right|^2$?

If $\tau \to 0$: $S_z \underbrace{\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}}_{-\frac{3}{4}A\hbar^2} = \frac{\hbar}{2} \underbrace{\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}}_{\frac{1}{4}A\hbar^2}$

Perturbation not important

$\tau < \frac{\cancel{\hbar}}{\cancel{\hbar}B} \sim \frac{1}{\hbar B}$.

If $\tau \to \infty$: adiabatic thm?

$1 \ll \frac{(B\hbar/2)^2}{\hbar \cdot \underbrace{\frac{d}{dt}\Delta E}}$ $\to \frac{B^2}{\hbar \min(A,B)}$

$\to \frac{1}{\tau}\min(\hbar^2 A, \hbar^2 B)$