PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 39

Berry phase

May 1

$\boxed{1}$ Adiabatic thm proof: if $H\{\lambda^\alpha(t)\} \rightarrow |\psi(0)\rangle = |\psi_n(\lambda^\alpha(0))\rangle$ if λ^α's vary slowly enough...

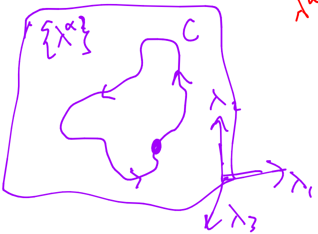
$$|\psi(t)\rangle = |\psi_n(\lambda^\alpha(t))\rangle \cdot \exp\left[-i\int_0^t dt' \frac{E_n(\lambda^\alpha(t'))}{\hbar}\right] \cdot \exp\Big[-i\underbrace{\int_0^t \sum_\alpha \cancel{dt} \frac{d\lambda^\alpha}{\cancel{dt}} A_\alpha}_{\gamma}\Big]$$

uninteresting / time-dep PT

$$A_\alpha = -i\langle\psi_n|\frac{\partial}{\partial\lambda^\alpha}|\psi_n\rangle$$

Berry connection

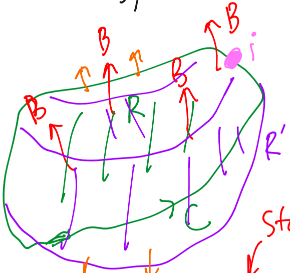
Berry phase: $\gamma = \int_{\lambda^\alpha(0)}^{\lambda^\alpha(t)} \sum_\alpha d\lambda^\alpha A_\alpha = \int d\vec{\lambda}\cdot\vec{A}$. independent of time t.



$$\gamma = \oint_C d\vec{\lambda}\cdot\vec{A} \rightarrow \neq 0 \text{ in general}$$

only $=0$ if $\vec{A} = \nabla\omega$

2 Analogy with magnetism: vector potential $\vec{A}$:



$\vec{B} = \nabla\times\vec{A}$.

gauge invariance

$\nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0$.

$\vec{A} \to \underline{\vec{A} + \boxed{\nabla w(x,y,z)}}$: leaves $\vec{B}$ invariant.

Stokes' Thm:

$$\oint_C d\vec{r}\cdot\vec{A} = \Phi_B = \int_R d\vec{s}\cdot\vec{B} = \text{magnetic flux}$$

$$= \int_{R'} d\vec{s}\cdot\vec{B} \qquad (\text{b/c } \nabla\cdot\vec{B} = 0)$$

$\vec{A}\to\vec{A}+\nabla w$:

$$\Phi_B \to \Phi_B + \underbrace{\oint_C d\vec{r}\cdot\nabla w}_{= \cancel{w(\vec{r}_i)} - \cancel{w(\vec{r}_i)}} \quad 0$$

Take R & R' together:

$$\int_R d\vec{s}\cdot\vec{B} + \int_{R'}(-d\vec{s})\cdot\vec{B} = \int_{\text{"sphere"}} d\vec{s}\cdot\vec{B} = \oint_{\text{bdy "sphere"}} d\vec{r}\cdot\vec{A} = 0.$$

$\boxed{3}$ Gauge invariance in Berry connection A_α?

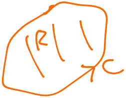
$$|\psi_n(\lambda^\alpha)\rangle \to e^{i\omega(\lambda)}\,|\psi_n(\lambda^\alpha)\rangle$$

$$\hookrightarrow A_\alpha = -i\langle\psi_n|\frac{\partial}{\partial\lambda^\alpha}|\psi_n\rangle \to A_\alpha + \frac{\partial\omega}{\partial\lambda^\alpha}$$

Berry phase $\gamma = \oint_C \sum_\alpha d\lambda^\alpha A_\alpha = \oint_C d\vec{\lambda}\cdot\vec{A}$ independent of ω!

Need to generalize $\vec{B}$ to higher d?

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial\lambda_\alpha} - \frac{\partial A_\alpha}{\partial\lambda_\beta} = -F_{\beta\alpha}$$



$\hookrightarrow$ n-dim: Stokes Thm:

$$\sum_\alpha \oint_C d\lambda^\alpha A_\alpha = \sum_{\alpha\beta}\int_\Omega dS^{\alpha\beta}F_{\alpha\beta} = \gamma$$

3 dimensions:

$$F = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}$$

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$\boxed{4}$ Example: $H = \vec{\lambda} \cdot \vec{S} = \cancel{\left(\frac{\hbar}{2}\right)} \begin{pmatrix} \lambda_z & \lambda_x - i\lambda_y \\ \lambda_x + i\lambda_y & -\lambda_z \end{pmatrix}$

spin-1/2 matrices

Parameterize: $(\lambda_x, \lambda_y, \lambda_z) = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)$

Eigenvalues of H?

$\det(H - E\mathbb{1}) = (E^2 - \lambda_z^2) - |\lambda_x + i\lambda_y|^2 \rightarrow E^2 - r^2 = 0.$ So $E = \pm r$.

$S_x S_y + S_y S_x = 0$ (for spin-1/2) etc.

$(S_x^2 = \mathbb{1})$

$H^2 = \lambda_x^2 \cancel{S_x^2} \mathbb{1} + \cancel{\lambda_x \lambda_y (S_x S_y + S_y S_x)} + \ldots$

$= (\lambda_x^2 + \lambda_y^2 + \lambda_z^2)\mathbb{1}$.

Eigenvectors:

$E = +r$

$|+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$

$E = -r$

$|-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$

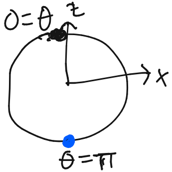
$\boxed{5}$

$E = +r$

$$|+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$E = -r$

$$|-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

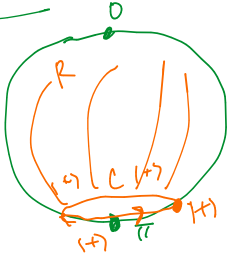


Are eigenstates well-defined? $[0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi]$

$\theta = \pi$: $\quad |+\rangle = \underbrace{e^{i\phi}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{}$ $\qquad |-\rangle = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

all same physical content ... but different math ...

No fix! $\qquad$ Look at $r=1$:



$$\gamma = \oint_C d\vec{x}\cdot\vec{A} = \int_R \sin\theta\, d\theta\, d\phi\, F_{\theta\phi}$$

$$F_{\theta\phi} = \frac{1}{2}.$$

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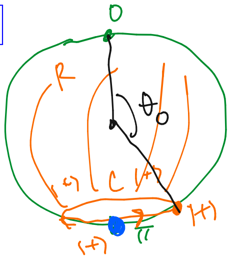

$$\gamma = \int_0^{\theta_0} \sin\theta \, d\theta \int_0^{2\pi} d\phi \cdot \frac{1}{2}$$

$$= \pi(1-\cos\theta_0)$$

$\theta_0 \to \pi$: loop C $\to$ point. $\gamma = 0$?

But $\gamma = 2\pi$. equal!

↳ what we need is $e^{i\gamma}$: $e^{i0} = 1 = e^{i2\pi}$

Berry flux through any "sphere" in 3-dim parameter space:

$$\int_{\text{sphere}} F = 2\pi \cdot n$$

[topology/geometry]

integer (Chern number)