PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 3X

Harmonic oscillator: series solution

January 25

1 Harmonic oscillator: $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

In QM: $p = -i\hbar \frac{d}{dx}$ so that $\underline{[x,p] = i\hbar}$

time-independent Schrödinger equation $H\psi = E\psi$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi = E\psi$$

Most solutions are not quantum states b/c ... not normalized

$$\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1 .$$

Need: $\psi(x) \to 0$ as $x \to \pm\infty$.

Like before: "dimensionless" units:

$$\left.\begin{aligned} \tilde{E} &= E/\hbar\omega \\ \tilde{x} &= x\sqrt{\frac{m\omega}{\hbar}} \end{aligned}\right\} \to \quad -\frac{1}{2}\frac{d^2\psi}{d\tilde{x}^2} + \frac{1}{2}\tilde{x}^2\psi = \tilde{E}\psi$$

From now, ignore $\sim$.

$\boxed{2}$

$$-\frac{1}{2}\underbrace{\frac{d^2\psi}{dx^2}}_{\psi''} + \frac{1}{2}x^2\psi = E\psi$$

express in terms of $x^k, e^x, \ldots$

A priori, no closed form solutions?

"much larger than"

Idea: approximate solns at large x: if $x^2 \gg 2E$:

$$\psi'' \approx x^2\psi$$

B/c ψ's Eqn is linear: $\psi = e^{f(x)}$.

$$\frac{d}{dx}\left(f' e^f\right) = f''\cancel{e^f} + f'^2\cancel{e^f} \approx x^2\cancel{e^f}$$

$$f'' + f'^2 = x^2 \quad \text{(at large } x\text{)}$$

$\boxed{3}$ $\overset{A}{f''} + \overset{B}{f'^2} = \overset{C}{x^2}$ $\qquad [\psi = e^f]$

Use method of dominant balance. If $0 = A + B + C$, then 2 largest terms $\approx$ same [up to const. factors]

$0 \neq 1 - 10^{-3} - 10^{-6}$ $\qquad$ "~~best~~" worst: $0 = 1 - 1/2 - 1/2$. $\left[1 - \frac{1}{3} - \frac{2}{3}\right]$

Often times we're lucky: $0 \approx 1 - 1.001 + 0.001$

$\approx 1 - 1.001$

Try: $f'' \approx x^2$. [large x]

Will f'^2 be small vs. x^2?

Since $f(x) \approx \frac{x^4}{12}$,

$f'^2 = \left(\frac{x^3}{3}\right)^2 = \frac{x^6}{9} \gg x^2$

Try: $f'^2 \approx x^2$

$f' = \pm x$

$f = \pm \frac{1}{2} x^2$ $\quad \Downarrow$

$f'' = \pm 1 \ll x^2$.

$f(x) \approx e^{+x^2/2}$ or $\underbrace{e^{-x^2/2}}$.

$\boxed{4}$ Summary: $\psi(x) \approx e^f \approx e^{-x^2/2}$

Goal: exact solution, peel off exponential: $\psi(x) = e^{-x^2/2} \cdot \underbrace{g(x)}_{\text{solve for.}}$

$$\frac{d\psi}{dx} = g' e^{-x^2/2} - x g e^{-x^2/2}$$

$$\frac{d^2\psi}{dx^2} = g'' e^{-x^2/2} - x g' e^{-x^2/2} - g e^{-x^2/2} - x g' e^{-x^2/2} + x^2 g e^{-x^2/2}$$

$$2E\psi = -\frac{d^2\psi}{dx^2} + x^2\psi$$

$$\left[2E \cdot g = -g'' + 2xg' + g\right] \cancel{e^{-x^2/2}}$$

$$0 = g'' - 2xg' + (2E-1)g$$

Apply dominant balance...

in general: [large x]

$$g'' \approx 2xg'$$

$$g' \approx e^{x^2}$$

$$g \sim \frac{e^{x^2}}{x} \rightarrow \psi \text{ not normalizable}$$

Must fail somehow...

$\boxed{5}$ $0 = g'' - 2xg' + (2E-1)g$. Solving for $g(x)$ & E.

$\Big\{$ Can use $g = x$: $0 = -2x \cdot 1 + (2E-1)x$ $\quad 2E-1 = 2$ or $E = 3/2 \Big\}$

or... $\underbrace{g = 1}$: $\quad 0 = 2E - 1$ or $\underbrace{E = 1/2}_{\text{ground state!}}$.

$\hookrightarrow \psi_0(x) = \# e^{-x^2/2}$ ✓

Thus: looking for special sol'ns @ special E.

Maybe $g(x)$ will = polynomial? [series solution].

Find g, then E [other order vs. lec. 2-3].

$\boxed{6}$ $0 = g'' - 2xg' + (2E-1)g.$

Try series for $g(x) = \sum_{n=0}^{\infty} c_n x^n$. Plug in;

$$0 = \sum_{n=0}^{\infty} c_n \left[n(n-1)x^{n-2} - 2x(nx^{n-1}) + (2E-1)x^n \right]$$

$$\sum c_n \left[n(n-1)x^{n-2} + (2E-1-2n)x^n \right]$$

$$0 = \sum_{n=0}^{\infty} x^n \Big[\underbrace{(n+1)(n+2)c_{n+2} + (2E-1-2n)c_n}_{=0} \Big]$$

$\left[c_{2n} \sim \frac{1}{n!} \right]$

$g(x) \approx e^{x^2}.$

True for all x: $c_{n+2} = -c_n \dfrac{-2n+2E-1}{(n+1)(n+2)} = c_n \dfrac{2n+1-2E}{(n+1)(n+2)}$

For general E: $c_0 = 1,\ c_2 = \cdots,\ c_4 = \cdots, \ldots$

or $c_1 = 1,\ c_3 = \cdots,$

large n:

$\dfrac{c_{n+2}}{c_n} \approx \dfrac{2}{n}$

$\boxed{7}$

$$c_{n+2} = -c_n \frac{-2n+2E-1}{(n+1)(n+2)} = c_n \frac{2n+1-2E}{(n+1)(n+2)}$$

Stop runaway $g \sim e^{x^2}$, $c_\ell = 0$ at large enough ℓ.

This happens if: for some k, $2k+1-2E=0$, can have $c_k \neq 0$

$c_{k+2}=0$

$c_{k+4}=0$

$\vdots$

series terminates at order k.

e.g. $E = 1/2$: $c_0 = 1$

$$c_2 = 1 \, \frac{\overset{0}{\cancel{2\cdot 0+1-1}}}{1\cdot 2} = 0.$$

Summary: If $E_n = n + \frac{1}{2}$, series for $g(x)$ terminates, $\psi(x)$ normalizable. (cf lec 5)

n^{th} exc. state

$$\psi_n(x) = \# \underbrace{H_n(x)}_{n^{th}\ \text{Hermite polynomial}} e^{-x^2/2}$$

n even: $H_n(x) = $ even

n odd: $H_n = $ odd