PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 4

Harmonic oscillator: higher dimensions

January 27

$\boxed{1}$ potential energy $V(\vec{x})$, local minimum $\vec{x}_0$.

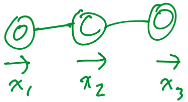
Close to $\vec{x}_0$: $\vec{x} = \vec{x}_0 + \delta\vec{x}$; $V(\vec{x}) = V(\vec{x}_0) + \frac{1}{2}\delta\vec{x}^T \underbrace{K}_{} \delta\vec{x}$

e-values positive

atoms in magnetic trap
(HW 2 Prob 3)

$V_{eff} \sim x^2 + y^2 + 8z^2$

molecular vibrations
(HW2 Prob 5)



$V_{eff} = \underbrace{(x_1 - x_2)}^2 + (x_2 - x_3)^2.$

Claim: (cf Class Mech) find normal modes coordinates

$$V_{eff} = \frac{m}{2}\left[\omega_1^2 X_1^2 + \omega_2^2 X_2^2 + \omega_3^2 X_3^2\right]$$

$\boxed{2}$ Focus on 2 dimensional system:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m\omega_x^2 x^2 + \frac{1}{2} m\omega_y^2 y^2$$

Particle in 2d: $\psi(x,y)$:

$$\underbrace{-\frac{\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial x^2}\right.}_{\partial/\partial x} + \underbrace{\left.\frac{\partial^2\psi}{\partial y^2}\right]}_{\partial/\partial y} + \frac{1}{2} m\Big(\underbrace{\omega_x^2 x^2}_{x} + \underbrace{\omega_y^2 y^2}_{y}\Big)\psi = E\psi.$$

Use separation of variables: $\psi(x,y) = X(x)Y(y)$

$$\left[-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2} + \frac{m\omega_x^2}{2}x^2 X\right]Y + X\left[-\frac{\hbar^2}{2m}\frac{d^2Y}{dy^2} + \frac{1}{2}m\omega_y^2 y^2 Y\right] = E\,XY$$

$$\underbrace{\left[-\frac{\hbar^2}{2m}\frac{X''}{X} + \frac{m\omega_x^2}{2}x^2\right]}_{\text{only dep. on } x \;=\text{ const. } E_x} + \underbrace{\left[-\frac{\hbar^2}{2m}\frac{Y''}{Y} + \frac{1}{2}m\omega_y^2 y^2\right]}_{\text{const. } E_y} = E$$

$$\underline{E = E_x + E_y}$$

$\boxed{3}$

$$-\frac{\hbar^2}{2m}\frac{X''}{X} + \frac{1}{2}m\omega_x^2 x^2 = E_x = \text{const.}$$

$$-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2} + \frac{1}{2}m\omega_x^2 x^2 X = E_x X$$ "=" 1d oscillator.

Thus $E_x = \hbar\omega_x(n_x + \frac{1}{2})$ $\quad \underbrace{n_x = 0, 1, 2, \ldots}_{\text{allowed energy levels}}$

$X(x) = \psi_n^{(\omega_x)}(x)$ from 1d,

Y equation same...

$$E = \hbar\omega_x(n_x + 1/2) + \hbar\omega_y(n_y + \tfrac{1}{2}) \qquad n_{x,y} = 0, 1, 2, \ldots$$

"non-interacting" system

implies ~~follows from~~ separability

$\boxed{4}$ Example: $\omega_x = \omega_y = \omega$. (isotropic oscillator)

What is ground state energy?

$$E = \hbar\omega\left(n_x + \frac{1}{2}\right) + \hbar\omega\left(n_y + \frac{1}{2}\right) = \underline{\hbar\omega(n_x + n_y + 1)} \qquad \overset{n_x\, n_y}{|0\,0\rangle}$$

$$= 2 \cdot \frac{1}{2}\hbar\omega \qquad [n_x = n_y = 0]$$

$$\psi(x,y) = \overbrace{\psi_0(x)\,\psi_0(y)}$$
$$= e^{-\left(\frac{\tilde{x}^2 + \tilde{y}^2}{2}\right)}$$

What's first excited state?

$E = 2\hbar\omega$: either $(n_x, n_y) = (1, 0)$ or $(0, 1)$

energy level is <u>degenerate</u> : 2 lin. ind. states obeying

$$H|\psi\rangle = 2\hbar\omega|\psi\rangle$$

$$\underbrace{|\psi\rangle = \alpha|01\rangle + \beta|10\rangle}$$

well-defined E = eigenstate of H = stationary?

$$|\psi(t)\rangle = \cancel{e^{-2i\omega t}}\,[\alpha|01\rangle + \beta|10\rangle].$$

5

Also solve oscillator by [More on lec 5.]

$$a_x = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega_x}{\hbar}}\,x + \frac{i p_x}{\sqrt{\hbar m \omega_x}}\right], \quad a_y \text{ similarly}$$

$$a_x |n_x n_y\rangle = \sqrt{n_x}\,|n_x - 1, n_y\rangle \qquad [a_x, a_y] = 0 \quad [a_x, a_y^\dagger] = 0.$$

If we have ℓ oscillators: $H = \sum_{j=1}^{\ell} \frac{p_j^2}{2m} + \frac{1}{2} m \omega_j^2 x_j^2$

energy levels $E = \sum_{j=1}^{\ell} \hbar\omega_j (n_j + 1/2)$.

e-states $|n_1 \cdots n_\ell\rangle$

$\psi_{n_1}(x_1)\,\psi_{n_2}(x_2) \cdots \psi_{n_\ell}(x_\ell)$

A 2d oscillator w/ no degeneracy:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \underbrace{\tfrac{1}{2} m\omega^2 x^2}_{\omega_x = \omega} + \tfrac{1}{2} m \underbrace{(2\omega^2)}_{\omega_y = \sqrt{2}\,\omega} y^2$$

$$E = \hbar\omega\left[\underline{n_x} + \frac{1}{2} + \underline{\sqrt{2}\,n_y} + \frac{\sqrt{2}}{2}\right].$$