PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 5

Multiple particles

January 30

$\boxed{1}$ Review of 2-level system:

$|\uparrow\rangle$ up $\qquad |\downarrow\rangle$ down. $\qquad$ General state: $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$

$$|a|^2 + |b|^2 = 1.$$

Define Pauli matrices:

$$\sigma_x = \begin{array}{cc} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{matrix} \uparrow \\ \downarrow \end{matrix} \\ \begin{matrix} \uparrow & \downarrow \end{matrix} & \end{array} \qquad S_x = \frac{\hbar}{2}\sigma_x \qquad \underline{\sigma_x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|}$$

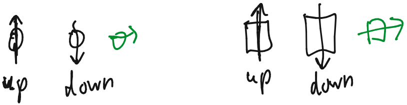
$$\sigma_x|\uparrow\rangle = |\downarrow\rangle$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\cancel{|\uparrow\rangle\langle\downarrow|\uparrow\rangle} + |\downarrow\rangle\cancelto{1}{\langle\uparrow|\uparrow\rangle}$$

$$\sigma_z = \begin{array}{c} \begin{matrix} \downarrow & \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{matrix} & \uparrow \end{matrix} \end{array} \qquad S_z = \frac{\hbar}{2}\sigma_z \qquad \sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$$

$\boxed{2}$ Universe of 2 particles:



Both particles exist. So allowed states:

$|\uparrow\uparrow\rangle \qquad |\uparrow\downarrow\rangle \qquad |\downarrow\uparrow\rangle \qquad |\downarrow\downarrow\rangle$

○ □

General wave: $|\psi\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle$

$|a|^2 + |b|^2 + |c|^2 + |d^2| = 1.$

If also $|○\rangle$, $|□\rangle$ allowed: how many total states?

[a 3rd state for each particle]

$9 = \underset{\text{states ○}}{3} \times \underset{\text{states □}}{3}$

$\boxed{3}$ I have two quantum systems:

($\mathcal{H}_A = \text{span}\{|a_1\rangle, \ldots, |a_n\rangle\}$)

vector spaces $\begin{cases} \mathcal{H}_A \leftarrow A: & |a_1\rangle, \ldots, |a_n\rangle \\ \mathcal{H}_B \leftarrow B: & |b_1\rangle, \ldots, |b_m\rangle \end{cases}$

combined AB: $|a_1 b_1\rangle \ldots |a_1 b_m\rangle \ldots |a_2 b_1\rangle \ldots \quad |a_n b_m\rangle$.

$|1\rangle \qquad\qquad |nm\rangle$

Define tensor product: $\underset{A}{|a_i\rangle} \otimes \underset{B}{|b_j\rangle} = |a_i b_j\rangle = |a_i\rangle|b_j\rangle$

$\mathcal{H}_A \otimes \mathcal{H}_B = \text{span}\{|a_1 b_1\rangle, \ldots |a_n b_m\rangle$
$|a_1\rangle \otimes b_1, \ldots, |a_n\rangle \otimes |b_m\rangle\}$

$\dim(\mathcal{H}_A \otimes \mathcal{H}_B)$
$= \dim(\mathcal{H}_A) \times \dim(\mathcal{H}_B)$.

$\boxed{4}$ What operator flips $\downarrow \leftrightarrow \uparrow$

Universe has 2 particle: $\sigma_{x,1}|\downarrow ?\rangle = |\uparrow ?\rangle$

Perspective 1:

$$\sigma_{x,1} = \begin{array}{c} \begin{array}{cccc} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \end{array} \\ \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \end{array}$$

visual guides

$$= \sigma_{x,1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \sigma_{x,1}|\uparrow\uparrow\rangle$$

Perspective 2: tensor product.

$$\sigma_{x,1}|s_1\rangle \otimes |s_2\rangle$$

$$= (\sigma_x |s_1\rangle) \otimes |s_2\rangle$$

So define

identity

$$\sigma_{x,1} = \underbrace{\sigma_x \otimes 1}$$

In general $C \otimes D$

$$(C \otimes D)|a\rangle \otimes |b\rangle$$

$$= (C|a\rangle) \otimes (D|b\rangle)$$

$\boxed{5}$ $H = \alpha\,\sigma_{z,1} + \beta\,\sigma_{z,2} = \alpha\,\sigma_z \otimes \mathbb{1} + \beta\,\mathbb{1} \otimes \sigma_z$

Let's write out H as a 4×4 matrix:

$$\alpha \begin{array}{c} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right) \\ \begin{array}{cccc} \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow \end{array} \end{array} + \beta \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right) = \underbrace{\begin{pmatrix} \alpha+\beta & 0 & 0 & 0 \\ 0 & \alpha-\beta & 0 & 0 \\ 0 & 0 & \beta-\alpha & 0 \\ 0 & 0 & 0 & -\alpha-\beta \end{pmatrix}}$$

$$\sigma_z \otimes \mathbb{1}\,|\uparrow\uparrow\rangle = \left(\sigma_z|\uparrow\rangle\right) \otimes \left(\mathbb{1}\,|\uparrow\rangle\right)$$
$$|\uparrow\rangle \quad \otimes \quad |\uparrow\rangle$$

eigenstates/vals:

$(\alpha+\beta)\,|\uparrow\uparrow\rangle = H|\uparrow\uparrow\rangle$

$(\alpha-\beta)\,|\uparrow\downarrow\rangle = H|\uparrow\downarrow\rangle$

$\vdots$

for spin 1:

$H = \alpha\,\sigma_z$:

$H|\uparrow\rangle = \alpha|\uparrow\rangle$

$H|\downarrow\rangle = -\alpha|\downarrow\rangle$

for spin 2:

$H = \beta\,\sigma_z$

$H|\uparrow\rangle = \beta|\uparrow\rangle$

$H|\downarrow\rangle = -\beta|\downarrow\rangle$

combined e-state/val:

$H|\uparrow\downarrow\rangle = \alpha|\uparrow\downarrow\rangle - \beta|\uparrow\downarrow\rangle$

$= (\alpha-\beta)\,|\uparrow\downarrow\rangle$

Separation of variables