PHYS 4410

Quantum Mechanics 2

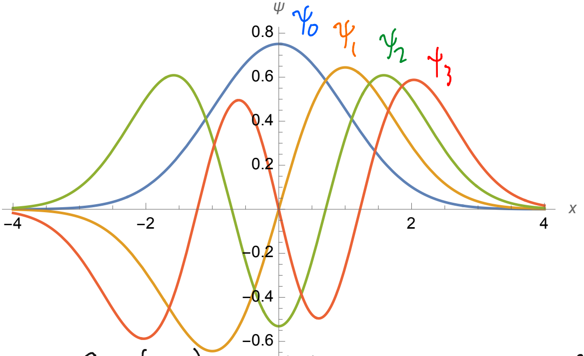
Spring 2023

Lecture 6

Parity symmetry

February 1

1 Here is a plot of the first few (unnormalized) eigenstates of the harmonic oscillator $\psi_n(x)$:



Pattern: $\psi_n(x) = \begin{cases} \psi_n(-x) & n \text{ even} \\ \underbrace{-\psi_n(-x)}_{\text{not invariant under } x \to -x \ldots} & n \text{ odd}. \end{cases}$

Why?

$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

invariant under $x \to -x$.

$\boxed{2}$ Symmetry = transformation leaving "equation of motion" the same / invariant.

In QM: $H|\psi\rangle = i\hbar \frac{d}{dt}|\psi\rangle$.

unitary matrix: $U^\dagger = U^{-1}$:

Reasonable transform? $|\psi\rangle \to U|\psi\rangle = |\phi\rangle$

$$\boxed{\begin{aligned}\langle\phi|\phi\rangle &= \underbrace{\langle\psi|U^\dagger}_{\langle\phi|}\underbrace{U|\psi\rangle}_{|\phi\rangle} \\ &= \langle\psi|U^{-1}U|\psi\rangle = \langle\psi|\psi\rangle\end{aligned}}$$

Assume U is ~~t~~-independent:

$$U\left[H|\psi\rangle = i\hbar\frac{d}{dt}|\psi\rangle\right]$$

$$\underbrace{UH|\psi\rangle} = i\hbar\frac{d}{dt}U|\psi\rangle = i\hbar\frac{d}{dt}|\phi\rangle$$

$$= UH\underbrace{\mathbb{1}}_{\text{identity}}|\psi\rangle = UHU^{-1}\underbrace{U|\psi\rangle}_{|\phi\rangle}$$

$$\underbrace{UHU^{-1}}_{\text{new Hamiltonian.}}|\phi\rangle = i\hbar\frac{d}{dt}|\phi\rangle$$

If $H = UHU^{-1}$, then U is a symmetry.

$\boxed{3}$ A symmetry (trans.) is unitary matrix U obeys $H = UHU^{-1}$. Depends on H.

Note: if $[H = UHU^{-1}]\,U$, so $HU = UH$
$[H, U] = 0$.

Claim 1: $U = \mathbb{1}$ is symmetry.

Claim 2: If U is symmetry, so is U^{-1}:

$$U^{-1}[H = UHU^{-1}]\,U$$

$$U^{-1}HU = \mathbb{1}H\mathbb{1} = \underbrace{H = U^{-1}HU}$$

Claim 3: If U_1 and U_2 are sym; so is $\underbrace{U_3 = U_1U_2}$.

$$U_1[U_2HU_2^{-1} = H]\,U_1^{-1}$$

$$U_1U_2H\underbrace{U_2^{-1}U_1^{-1}} = U_1HU_1^{-1}$$

$$U_3H\underbrace{U_3^{-1}} = H$$

These imply the set of symmetries U form a mathematical group.

$\boxed{4}$ Claim: harmonic oscillator has parity symmetry.

$x \to -x$. In QM: $P\,\psi(x) = \psi(-x)$.

Since $P^2 \cdot \psi(x) = P \cdot \psi(-x) = \psi[-(-x)] = \psi(x) = 1 \cdot \psi(x)$

$\{1, P\}$ form a group. [$\mathbb{Z}_2$ symmetry].

Claim: for harmonic oscillator $[P, H] = 0$ or $PHP^{-1} = H$

$PHP = H$

$HP\,\psi(x) \stackrel{?}{=} PH\,\psi$

$$\frac{d\psi(-x)}{dx} = -\psi'(-x)$$

[chain rule]

$$HP\,\psi(x) = H\psi(-x) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2}\right]\psi(-x)$$

$$= -\frac{\hbar^2}{2m}\underbrace{\psi''[-x]}_{\left.\frac{d^2\psi[x]}{dx^2}\right|_{x\to -x}} + \frac{m\omega^2}{2}(-x)^2\,\psi(-x)$$

$$= \left(H\psi[x]\right)\Big|_{x \to -x} = PH\psi(x).$$

$\boxed{5}$ Theorem: if $[A,B]=0$, there's a basis where A and B are both diagonal.

Proof sketch: let $|a\rangle$ be e-vector of A:

$$A|a\rangle = a|a\rangle$$

Now: $[A,B]|a\rangle = 0|a\rangle$

$$A(B|a\rangle) - \underbrace{B(A|a\rangle)}_{aB|a\rangle}$$

Therefore $A(B|a\rangle) = a(B|a\rangle)$

so $B|a\rangle$ also has e-value a...

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Back to oscillator: $[H, P] = 0 = [P, H]$.

Know basis $|n\rangle$ where H is diagonal:

$$H|n\rangle = \hbar\omega(n + 1/2)|n\rangle$$

Thm: each $|n\rangle$ must be eigenvector of P.

Since $P^2 = 1$, let's imagine $|\psi\rangle$ is e-vector of P:

$$P|\psi\rangle = \lambda|\psi\rangle$$

$$1|\psi\rangle = P^2|\psi\rangle = \lambda^2|\psi\rangle = |\psi\rangle, \quad \text{so } \lambda^2 = 1 \text{ or } \lambda = \pm 1$$

in some basis

$$P = \left(\begin{array}{cccc|ccc} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ \hline & & & & -1 & & \\ & & & & & -1 & \\ & & & & & & -1 \end{array}\right)\begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{even representation: } P\psi(x) = \psi(-x) = \psi(x) \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{odd representation: } P\psi(x) = \psi(-x) = -\psi(x) \end{array}$$

diagonal

$\boxed{7}$ Which $|n\rangle$ are even vs. odd?

Recall: $\psi_0(x) = e^{-\tilde{x}^2/2} = \psi_0(-x)$. So $|0\rangle$ is even.

$$a^\dagger \psi(x) = \left(x - \frac{d}{dx}\right)\psi(x)$$

schematic

If $\psi(x)$ is even: $\underbrace{x\,\psi(x)}_{\text{odd}} \xrightarrow{P} (-x)\,\psi(-x) = -x\,\psi(x)$

Similarly, $\frac{d\psi}{dx}$ odd...

$$\underset{\text{even}}{|0\rangle} \xrightarrow{a^\dagger} \underset{\text{odd}}{|1\rangle} \xrightarrow{a^\dagger} \underset{\text{even}}{|2\rangle} \longrightarrow \cdots$$