PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 7

Bosons and fermions

February 3

$\boxed{1}$ let H_0 describe a single quantum particle: $H_0|E\rangle = E|E\rangle$

e-state/vector   e-value

Now suppose I have two <u>identical</u> particles in the same "potential":

$$H = \underbrace{H_0 \otimes 1 + 1 \otimes H_0}_{\text{non-interacting particles}}$$

common shorthand: $\underbrace{H_{0,1}}_{\text{act on particle 1}} + \underbrace{H_{0,2}}_{\text{act on particle 2}}$

Lecture 5: The two particle e-states/e-values are:

$$H\left(|E_1\rangle \otimes |E_2\rangle\right) = \left(H_0 \otimes 1 + 1 \otimes H_0\right)|E_1\rangle \otimes |E_2\rangle$$

$$= \left(H_0|E_1\rangle\right) \otimes |E_2\rangle + |E_1\rangle \otimes \left(H_0|E_2\rangle\right)$$

$$= (E_1 + E_2)\underbrace{|E_1\rangle \otimes |E_2\rangle}_{}$$

common shorthand:

$|E_1 E_2\rangle$.   or: $|E_1\rangle|E_2\rangle$

$\boxed{2}$ Our 2 particles have a "parity" ($\mathbb{Z}_2$) symmetry:

let $P|E_1E_2\rangle = |E_2E_1\rangle$. P = "particle exchange".

Claim: $[H, P] = 0$. Check: $[H,P]|E_1E_2\rangle \stackrel{?}{=} 0$ for any E_1E_2?

$$HP|E_1E_2\rangle = H|E_2E_1\rangle = (E_2+E_1)|E_2E_1\rangle \quad \checkmark$$

$$PH|E_1E_2\rangle = P\big((E_1+E_2)|E_1E_2\rangle\big) = \underbrace{(E_1+E_2)}_{\text{const.}}P|E_1E_2\rangle = (E_1+E_2)|E_2E_1\rangle$$

Claim: $P^2 = 1$. Check: $P^2|E_1E_2\rangle = P|E_2E_1\rangle = |E_1E_2\rangle$ ✓

Therefore, e-states of H can be classified as either

<u>even</u> or <u>odd</u> under P.

$\boxed{3}$ Claim: for any state $|\psi\rangle$, $(\mathbb{1} \pm P)|\psi\rangle$ is an eigenvector of P w/ eigenvalue ± 1.

Check: $P(\mathbb{1} \pm P)|\psi\rangle = (P \pm \mathbb{1})|\psi\rangle = \pm(\mathbb{1} \pm P)|\psi\rangle$ ✓

Define even $(+1)$ states to be <u>bosonic</u>. What are they?

Try $|\psi\rangle = |E_1 E_1\rangle$. $\quad (\mathbb{1}+P)|\psi\rangle = |E_1 E_1\rangle + |E_1 E_1\rangle = 2|E_1 E_1\rangle$.

same state!

Normalize: $\begin{cases} H|E_1 E_1\rangle = 2E_1|E_1 E_1\rangle \\ P|E_1 E_1\rangle = +1\,|E_1 E_1\rangle \end{cases}$.

diff. states

Try $|\psi\rangle = |E_1 E_2\rangle$. $\quad (\mathbb{1}+P)|\psi\rangle = |E_1 E_2\rangle + |E_2 E_1\rangle$.

Normalize: $H\left(\frac{|E_1 E_2\rangle + |E_2 E_1\rangle}{\sqrt{2}}\right) = (E_1+E_2)\frac{|E_1 E_2\rangle + |E_2 E_1\rangle}{\sqrt{2}}$

Define odd (-1) states to be <u>fermionic</u>.

Try $|\psi\rangle = |E_1 E_2\rangle$ .... get $H\left(\frac{|E_1 E_2\rangle - |E_2 E_1\rangle}{\sqrt{2}}\right) = (E_1+E_2)\frac{|E_1 E_2\rangle - |E_2 E_1\rangle}{\sqrt{2}}$

Try $|\psi\rangle = |E_1 E_1\rangle$? $\quad (\mathbb{1}-P)|E_1 E_1\rangle = |E_1 E_1\rangle - |E_1 E_1\rangle = 0$.

<u>Pauli exclusion</u>

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Suppose 2 indistinguishable particles...

$$H_0 = \begin{array}{c} \begin{array}{ccc} |1\rangle & |2\rangle & |3\rangle \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 2\varepsilon \end{pmatrix} \end{array} \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$H_0 = \varepsilon|2\rangle\langle 2| + 2\varepsilon|3\rangle\langle 3|.$$

If particles are bosons: what is ground state?

$H|11\rangle = 0.$ Put both particles in lowest energy state!

How many bosonic states in Hilbert space?

$|11\rangle, |22\rangle, |33\rangle$: 3 states occupying same state.

$\frac{|12\rangle + |21\rangle}{\sqrt{2}}$, or 13, or 23: 3 states. $3+3=\underline{6 \text{ total}}$

If particles are fermions: what's ground state?

$\frac{|12\rangle - |21\rangle}{\sqrt{2}}$: Pauli exclusion forbids both from being in $|1\rangle$!

$\boxed{5}$ As we'll see later, particles have <u>spin</u>: $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

Relativity + QM: bosons: $s = 0, 1, 2, \ldots$
fermions: $s = 1/2, 3/2, \ldots$

For now: just know $s=0$, $s = 1/2$.

no internal spin DOF

spin in either $|\uparrow\rangle$ or $|\downarrow\rangle$.

For each particle in system, $\otimes$ $\begin{cases} |\uparrow\rangle \\ |\downarrow\rangle \end{cases}$ to Hilbert space:

$$|E_1\rangle \otimes |E_2\rangle \longrightarrow |E_1 \underset{\text{or}\downarrow}{\uparrow}\rangle \otimes |E_2 \underset{\text{or}\downarrow}{\uparrow}\rangle$$

Electron, proton, neutron...: all spin-1/2 fermions!

Particle exchange $P|E_1 \underset{\text{spin}}{s_1}\rangle \otimes |E_2 s_2\rangle = \underbrace{|E_2 s_2\rangle \otimes |E_1 s_1\rangle}_{\text{all internal information swaps!}}$

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Suppose 2 indistinguishable spin-1/2 particles

often implicit: "the non-spin degree of freedom has energy levels..."

$$H_0 = \begin{array}{c} \begin{array}{ccc} |1\rangle & |2\rangle & |3\rangle \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 2\varepsilon \end{pmatrix} \end{array} \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$H_0 = \varepsilon|2\rangle\langle 2| + 2\varepsilon|3\rangle\langle 3|.$$

What is the ground state?

Now by Pauli exclusion, we can put fermions in $|1\uparrow\rangle$ and $|1\downarrow\rangle$:

$$|\psi\rangle = \frac{|1\uparrow\rangle \otimes |1\downarrow\rangle - |1\downarrow\rangle \otimes |1\uparrow\rangle}{\sqrt{2}}.$$

$H|\psi\rangle = 0$.

note: now writing $|E_1 E_2\rangle|S_1 S_2\rangle$

Often useful to factor: $|\psi\rangle = \underbrace{|11\rangle}_{\text{symmetric "position space" wave function?}} \otimes \underbrace{\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}}_{\text{antisymmetric spin wave function.}}$