PHYS 4410

Quantum Mechanics 2

Spring 2023

Lecture 8

Three or more bosons or fermions

February 6

$\boxed{1}$ Review Lec 7: particle exchange $P_{12}\underbrace{|E_1E_2\rangle}_{|E_1\rangle\otimes|E_2\rangle} = |E_2E_1\rangle$

Since $P_{12}^2 = \mathbb{1}$ (identity), classify $|\psi\rangle$'s as even/odd:

Even states: boson. $P_{12}|\psi\rangle = |\psi\rangle$

- $|E_1E_1\rangle$
- $\dfrac{|E_1E_2\rangle + |E_2E_1\rangle}{\sqrt{2}}$ ← different (orthogonal) states

if spin:

$|E_1, E_2\rangle$

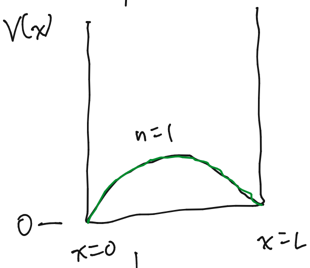
$\downarrow P_{12}$

$|E_1 S_1, E_2 S_2\rangle$

Odd states: fermionic $P_{12}|\psi\rangle = -|\psi\rangle$.

- $\dfrac{|E_1E_2\rangle - |E_2E_1\rangle}{\sqrt{2}}$
- Pauli exclusion: 2 fermions must be in diff. states

$\boxed{2}$ Consider particle(s) in infinite sq well:



$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

$n = 1, 2, 3, \ldots$

$$\psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

Put in: 2 $\underbrace{\text{spin-0}}$ particles.

$\underline{\text{boson}}$: "no" internal spin state

$$H = H_{ISW} \otimes 1 + 1 \otimes H_{ISW}$$

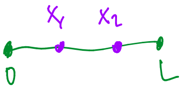
Assume $\underbrace{\text{non-interacting}}$:

Ground state energy:

$$E_{gs} = E_1 + E_1 = 2E_1$$

$$|gs\rangle = |1,1\rangle = |1\rangle \otimes |1\rangle$$

$$\psi(x_1, x_2) = \psi_{n=1}(x_1) \cdot \psi_{n=1}(x_2)$$



$$\int_0^L dx_1 \int_0^L dx_2 \, |\psi(x_1, x_2)|^2 = 1.$$

$\boxed{3}$ Now: 2 spin-1/2 fermions.

Ground state?

$E = E_1 + E_1 = 2E_1$

$$\frac{|\underbrace{1\uparrow}_{n_1 s_1}, \underbrace{1\downarrow}_{n_2 s_2}\rangle - |1\downarrow, 1\uparrow\rangle}{\sqrt{2}}$$

$P_{12}|n_1 s_1, n_2 s_2\rangle = |n_2 s_2, n_1 s_1\rangle$

Find odd state:

even under $n_1 \leftrightarrow n_2$ / odd $s_1 \leftrightarrow s_2$

odd $n_1 \leftrightarrow n_2$ / even $s_1 \leftrightarrow s_2$

$$\underbrace{|11\rangle}_{n_1 n_2} \otimes \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$$

+1   −1   under $P_{12} \to -1$

First excited state?

$n_1 = 2 \quad n_2 = 1$

$s_1 = \uparrow \quad s_2 = \downarrow$

$$\frac{|2\uparrow, 1\downarrow\rangle - |1\downarrow, 2\uparrow\rangle}{\sqrt{2}}, \text{ or } \frac{|2\downarrow, 1\uparrow\rangle - |1\uparrow, 2\downarrow\rangle}{\sqrt{2}}$$

$$\frac{|2\uparrow, 1\uparrow\rangle - |1\uparrow, 2\uparrow\rangle}{\sqrt{2}} \text{ or } \frac{|2\downarrow, 1\downarrow\rangle - |1\downarrow, 2\downarrow\rangle}{\sqrt{2}}$$

degeneracy of 4

$\boxed{4}$ $k > 2$ indistinguishable particles.

define $\underbrace{P_{ij}}_{\text{exchange for } i \,\&\, j} |\psi_1 \cdots \psi_i \cdots \psi_j \cdots \psi_k\rangle = \underbrace{|\psi_1 \cdots \psi_j \cdots \psi_i \cdots \psi_k\rangle}_{\text{e.g. } \Psi_{tot}}$

single particle (e.g. n, s_1)

$k=3$ example:

$P_{12}|abc\rangle = |bac\rangle$

$P_{13}|abc\rangle = |cba\rangle$

$P_{23}|abc\rangle = |acb\rangle$

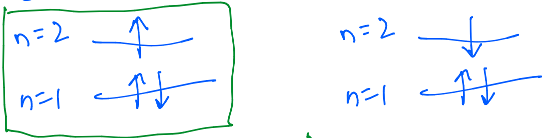
For k bosons: $P_{ij}|\Psi_{tot}\rangle = |\Psi_{tot}\rangle$ for any i & j.

For k fermions: $P_{ij}|\Psi_{tot}\rangle = -|\Psi_{tot}\rangle$

5 Let's add 3^{rd} spin-$\frac{1}{2}$ particle. <u>fermions</u>

Ground state?

$E = 2E_1 + E_2$



degenerate 2 states

Slater determinant:

$$|\psi_{tot}\rangle = \frac{1}{\sqrt{3!}}\Big[\,|1\uparrow, 1\downarrow, 2\uparrow\rangle - |1\downarrow, 1\uparrow, 2\uparrow\rangle - |2\uparrow, 1\downarrow, 1\uparrow\rangle - |1\uparrow, 2\uparrow, 1\downarrow\rangle + |1\downarrow, 2\uparrow, 1\uparrow\rangle + |2\uparrow, 1\uparrow, 1\downarrow\rangle\Big]$$

$\boxed{6}$ Back to spin-0 bosons: 3 particles

Ground state: $E = 3E_1$ $\quad |1,1,1\rangle$

First excited state: $E = E_1 + E_1 + E_2$

$$|\psi\rangle \sim C\Big[\underset{\checkmark}{|112\rangle} + \underset{\checkmark P_{12}}{|112\rangle} + \underset{P_{13}}{|211\rangle} + |211\rangle + \underset{P_{23}}{|121\rangle} + |121\rangle\Big]$$

$$\sim \frac{1}{\sqrt{3}}\Big[|112\rangle + |121\rangle + |211\rangle\Big]$$

"$\sim |2$ bosons in $n=1$, 1 boson in $n=2\rangle$."