PHYS 4410: Quantum Mechanics 2 **Spring 2023**

Practice Exam 1

20 **Problem 1:** Consider 3 indistinguishable spin-0 particles of mass m in a one-dimensional infinite square well ("particle in a box") of length L. Assume that the particles are non-interacting.

1. What is the ground state energy of the system? What is the ground state wave function $\psi(x_1, x_2, x_3)$? Here $x_{1,2,3}$ denote the positions of the three particles.

2. What is the energy of the first excited state of the system? Write down all linearly independent eigenfunctions of the Hamiltonian with this energy.

15 **Problem 2:** Consider the following two-dimensional harmonic oscillator:

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2}m\omega^2 x_1^2 + 2m\omega^2 x_2^2. \tag{1}$$

1. What are the eigenvalues of H?

2. What is the degeneracy of the lowest five energy levels?

Problem 3: Consider the quantum harmonic oscillator. You may work in units where $m = \omega = \hbar = 1$.

15 **A:** Let $|\psi\rangle$ be a state that obeys $H|\psi\rangle = \frac{5}{2}\hbar\omega|\psi\rangle$. For this part only, don't worry about normalizing $|\psi\rangle$.

- A1. Describe how to calculate $|\psi\rangle$ (make sure it is normalized!) using raising operators.
- A2. Give an expression for the position-space wave function $\psi(x)$.

15 **B:** Continue to work with the state $|\psi\rangle$.

- B1. Calculate Δx in this state.
- B2. Calculate Δp in this state.
- B3. Check that the Heisenberg uncertainty principle is obeyed.

Problem 4: Consider two particles, each of which can occupy two distinct quantum states. We will call (for one particle) the two states $|\uparrow\rangle$ and $|\downarrow\rangle$, although for this problem we will *not* associate these with the up/down states of a physical spin-$\frac{1}{2}$ particle. Consider the Hamiltonian

$$H = A\left(\sigma_z \otimes 1 + 1 \otimes \sigma_z\right) + B\sigma_x \otimes \sigma_x. \tag{2}$$

Here we are denoting $\sigma_x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|$ and $\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$.

10 **A:** Write out H as a 4×4 matrix. Be sure to carefully explain what the states are in Hilbert space and which row/column of H corresponds to which state.

15 **B:** Now, let us find the eigenvalues of H.

- B1. Show that $[H, \sigma_z \otimes \sigma_z] = 0$.
- B2. Use this fact to simplify the calculation of the eigenvalues of H down to diagonalizing two different 2×2 matrices. Report the 4 different eigenvalues of H.

10 **C:** Now, consider the particle exchange operator P, defined so that $P|s_1\rangle \otimes |s_2\rangle = |s_2\rangle \otimes |s_1\rangle$. Here $s_{1,2} = \uparrow, \downarrow$ denote the possible states of the two particles.

- C1. Show that $[H, P] = 0$.
- C2. Describe how the eigenvectors of H will decompose into even and odd states under particle exchange.
- C3. If the two-level system associated to each particle was genuinely its spin, would the particle be a boson or a fermion?
- C4. Based on your answer to C3, deduce the possible states that the particle could occupy.

Problem 5 (Coherent states): Coherent states $|\alpha\rangle$ of the quantum harmonic oscillator are defined as eigenstates of the lowering operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{3}$$

Here α is a complex number. Coherent states have a lot of practical uses in quantum optics, e.g.

15 **A:** Let us look for an explicit expression for a coherent state.

- A1. Suppose that

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n(\alpha)|n\rangle, \tag{4}$$

  where $c_n(\alpha)$ are unknown coefficients. Use (3) to find expressions for $c_n(\alpha)$ for all n. You do not need to normalize the c_n.
- A2. Show that a time-evolved coherent state remains a coherent state. Give a formula for $\alpha(t)$, the coherent state's eigenvalue, as a function of time t.

The fact that coherent states remain coherent makes them the "most classical" of all the states of the quantum harmonic oscillator.

5 **B:** Could there be an analogue of coherent states where we define $a^\dagger|\beta\rangle = \beta|\beta\rangle$? Why or why not?