PHYS 4410: Quantum Mechanics 2 **Spring 2023**

Practice Exam 2

Problem 1: Consider 2 spin-1 particles, with spin angular momenta $\mathbf{S}_{1,2}$.

5 **A:** Are these particles bosons or fermions?

10 **B:** Evaluate $1 \otimes 1$, and explain what this notation represents.

15 **C:** Let $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ denote the total spin. In terms of the uncoupled basis $|1m_1 1m_2\rangle$, write the coupled basis state $|s = 1, m = 1\rangle$.

15 **D:** Consider these two particles placed into a one-dimensional harmonic oscillator potential. You can set $\hbar = m = \omega = 1$ if you like.

- D1. What is the ground state energy, and what is the degeneracy of the state?
- D2. What is the first excited state, and what is its degeneracy?

Problem 2: Consider two spin-$\frac{1}{2}$ fermions on a ring, with Hamiltonian

$$H = -A\frac{\partial^2}{\partial\phi_1^2} - A\frac{\partial^2}{\partial\phi_2^2} + \left\{ \begin{array}{ll} B & |\phi_1 - \phi_2| \le \alpha \\ 0 & \text{otherwise} \end{array} \right. . \tag{1}$$

Here $A, B > 0$ are constants, and $\alpha \ll 1$ is a positive constant.

10 **A:** First consider the problem when $B = 0$. Let $|n_1 n_2\rangle$ denote the wave function $(2\pi)^{-1}\mathrm{e}^{\mathrm{i}(n_1\phi_1 + n_2\phi_2)}$.

- A1. Explain why the unique ground state is

$$|\mathrm{g}\rangle = |00\rangle \otimes \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}. \tag{2}$$

- A2. Show that one of the first-excited states is the following state from the spin triplet:

$$|\mathrm{e}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \otimes |\uparrow\uparrow\rangle. \tag{3}$$

15 **B:** Now, consider the theory with $B \neq 0$.

- B1. Evaluate $\langle \mathrm{g}|H|\mathrm{g}\rangle$ and $\langle \mathrm{e}|H|\mathrm{e}\rangle$.
- B2. Describe the likely spin of the ground state of this model as a function of B.

15 **Problem 3 (Landau levels):** Consider an electron of mass m and charge $-e$, moving in a very thin layer of material, such that the motion of the electron can be approximated as two-dimensional. A magnetic field of strength B is applied perpendicular to the direction of motion. The Hamiltonian for an electron moving in this material can be chosen as

$$H = \frac{p_x^2}{2m} + \frac{(p_y + eBx)^2}{2m}. \tag{4}$$

1. Show that $[H, p_y] = 0$.
2. Show that the eigenvalues of H are[1]

$$E_n = \hbar \frac{eB}{m}\left(n + \frac{1}{2}\right), \quad (n = 0, 1, 2, \ldots). \tag{5}$$

   These wave functions are called **Landau levels**.

3. In condensed matter physics, it is very interesting to study electrons in the lowest Landau level. Using effective electron mass $m \approx 10^{-31}$ kg, laboratory magnetic field strength $B \sim 1$ T, $e = 1.6 \times 10^{-19}$ C and $\hbar \approx 10^{-34}$ J $\cdot$ s, estimate the energy gap between the $n = 0$ and $n = 1$ states. Compare your answer to 10^{-23} J, the energy scale associated to a temperature of 1 K (which is readily achievable in the lab).

Problem 4: Consider a three-level quantum system with Hamiltonian

$$H = -h\mathrm{e}^{\mathrm{i}\phi}R - h\mathrm{e}^{-\mathrm{i}\phi}R^2, \tag{6}$$

where h and ϕ are real constants, while

$$R = |2\rangle\langle 1| + |3\rangle\langle 2| + |1\rangle\langle 3|. \tag{7}$$

15 **A:** Let us begin by exactly finding the eigenvalues of H.

- A1. Show that $R^3 = 1$.
- A2. Show that $[H, R] = 0$. Thus deduce H has a $\mathbb{Z}_3$ symmetry generated by R.
- A3. Use the $\mathbb{Z}_3$ symmetry to find the eigenvalues and eigenvectors of H.

10 **B:** Notice that if $\phi = 0$, H becomes degenerate. This is not accidental.

- B1. Show that when $\phi = 0$, H has a $\mathbb{Z}_2$ symmetry generated by

$$S = |1\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|. \tag{8}$$

- B2. What is the full symmetry group of H?
- B3. Explain why the Hilbert space consists of (the direct sum of) a one-dimensional and two-dimensional irreducible representation of the symmetry group.
- B4. Deduce the origin of the degeneracy of H.

10 **Problem 5:** Let $A, B \neq 0$ be generic constants. Let $\mathbf{S}_{1,2,3,4}$ denote the spin matrices acting on four *distinguishable* particles, each of spin-$\frac{1}{2}$. Find the eigenvalues (and their degeneracies) of the Hamiltonian

$$H = A\mathbf{S}_1 \cdot \mathbf{S}_2 + A\mathbf{S}_3 \cdot \mathbf{S}_4 + B(\mathbf{S}_1 + \mathbf{S}_2) \cdot (\mathbf{S}_3 + \mathbf{S}_4). \tag{9}$$

[1] *Hint:* If you define $\tilde{x} = x + c$ for any constant c, then $[\tilde{x}, p_x] = \mathrm{i}\hbar$ continues to hold.