PHYS 4410: Quantum Mechanics 2 **Spring 2023**

Practice Exam 3

Problem 1: Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + gx^4, \tag{1}$$

where g is a small parameter.

20 **A:** Use time-independent perturbation theory to predict the eigenvalues of H, up to first order in the small parameter g.

5 **B:** At what energy scale will perturbation theory fail to be accurate?[1]

20 **C:** For large energies compared to this energy scale, we can use the Bohr-Sommerfeld approximation. Show that for very large n,

$$E_n \approx g\left[\frac{\pi\hbar n}{C\sqrt{2mg}}\right]^{4/3}, \tag{2}$$

where the dimensionless number

$$C = \int_{-1}^{1} \mathrm{d}z\, \sqrt{1-z^4}. \tag{3}$$

Problem 2: Consider a single particle moving on a ring, with Hamiltonian

$$H = -A\frac{\partial^2}{\partial\phi^2} - B\cos\phi, \tag{4}$$

Here A and B are positive constants. $\phi \sim \phi + 2\pi$ is an angular coordinate.

15 **A:** Suppose that $B \ll A$. Calculate the ground state energy of H to quadratic order in B.

5 **B:** Now suppose that $B \gg A$. Can you estimate the ground state energy of H in this limit?

Problem 3: Consider a time-dependent Hamiltonian for a spin-$\frac{1}{2}$ particle:

$$H(t) = AS_z + B\left(S_x - S_z\right)\mathrm{e}^{-|t|/\tau}. \tag{5}$$

Let $A, B > 0$ be constants. Suppose the wave function obeys $|\psi(t\to-\infty)\rangle \sim |\downarrow\rangle$ (up to an unimportant phase factor).

20 **A:** Begin by assuming that $B \ll A$, so that we can use time-dependent perturbation theory. Calculate the probability that the perturbation will induce a transition to $|\uparrow\rangle$ at time t: i.e. $|\langle\uparrow|\psi(t)\rangle|^2$.

[1] *Hint:* Compare the size of the "small" perturbative correction to the leading order energy, as a function of the energy level n of the harmonic oscillator.

10 **B:** Now suppose that $B = A$, such that time-dependent perturbation theory can no longer be applied. While you do not need to find a exact solution for $|\psi(t)\rangle$, under what criteria on τ will we find $|\langle\downarrow|\psi(+\infty)\rangle|^2 \approx 1$?

15 **Problem 4 (Spin-orbit interactions):** In this problem, we will consider a more microscopic model for the hyperfine interaction – and more generally what are called **spin-orbit interactions** – in rotationally invariant problems. Let us focus on the non-relativistic hydrogen atom. In this case, we have

$$H = \frac{\mathbf{p}^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r} + \frac{\delta}{r^3}\mathbf{L}\cdot\mathbf{S}, \tag{6}$$

The spin of the electron is decoupled from the orbital problem (at $\delta = 0$).

Now consider the case $\delta \neq 0$, but treat δ as a small parameter. Within first-order (degenerate) perturbation theory, describe how the energy levels of the hydrogen atom will be modified. You do not need to evaluate any integrals in this problem, and can assume that (unless an integral must be zero) any distinct integrals you encounter in this problem have different numerical values.

10 **Problem 5:** Consider a one-dimensional particle with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{A}{(x^2+a^2)^{\alpha/2}}. \tag{7}$$

Argue that there is a critical value of α, called α_*, such that for $\alpha < \alpha_*$ there are infinitely many bound states of H (eigenvalues < 0), while for $\alpha > \alpha_*$ there are only a finite number of bound states. As part of your answer, you should explain the value of α_*.