PHYS 5040: Algebra and Topology in Physics **Spring 2021**

Exam

- **Due:** 11:59 PM, May 4. Submit electronically on Canvas.
- You are allowed to use any course materials (including posted solutions), any books, and online references such as Wikipedia for help on this exam. **Do not collaborate** with any human, or ask for help via PhysicsForums, Chegg, Quora or any similar website. You may ask the instructor alone for help in the form of clarifying questions.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

20 **Problem 1:** Consider the Lie group SU(5).

1.1. In high energy notation, SU(5) has irreps $\mathbf{1}$, $\mathbf{5}$, $\bar{\mathbf{5}}$ and $\mathbf{24}$. What tensors/vectors do each of these irreps correspond to?

1.2. Find a subgroup of SU(5) isomorphic to SO(5), in which the $\mathbf{5}$ and $\bar{\mathbf{5}}$ irreps of SU(5) both become the *same* 5-dimensional irrep of SO(5) (which is often also denoted as $\mathbf{5}$).[1]

1.3. Decompose the $\mathbf{1}$ and $\mathbf{24}$ of SU(5) into irreps of the SO(5) subgroup.

20 **Problem 2 (Trigonal bipyramidal molecule):** Consider the molecule sketched in Figure 1, whose shape is described as trigonal bipyramidal. An example of such a molecule is PF_5.

2.1. Let G denote the symmetry group of PF_5. Since there are 6 atoms, we might aim to describe G as a subgroup of S_6. Write down a set of *generators* for all of the permutations in S_6 that generate the group G, such as (12).

2.2. Show that $G = S_3 \times \mathbb{Z}_2$.

2.3. Argue that we expect generic degeneracies in the electronic Hamiltonian in this molecule.

2.4. Argue that the subgroup $K \leq G$, which corresponds to *rotations* of the PF_5 molecule in space, without any reflections, is isomorphic to S_3.

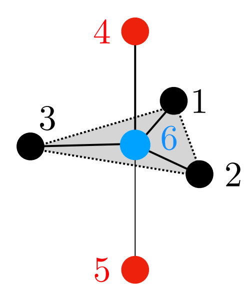

Figure 1: A trigonal bipyramidal molecule such as PF_5. Black F atoms 1-3 are at the corners of an equilateral triangle in the xy-plane; red F atoms 4 and 5 sit along the z-axis above P atom 6, which is located at the origin.

[1] *Hint:* If $g \in \mathrm{SU}(5)$, think about the relationship between the 5×5 matrices $U(g)$ in the irrep $\mathbf{5}$, as well as in the irrep $\bar{\mathbf{5}}$. When do these matrices equal each other?

20 **Problem 3:** Let A, B, C, D and E denote 5 non-homeomorphic topological spaces. As a "puzzle game", I tell you that

$$\{A, B, C, D, E\} = \{\mathbb{RP}^2, \mathrm{SU}(2), \mathrm{S}^1, \mathrm{S}^6, \mathbb{R}^6\}, \tag{1}$$

and that

$$\pi_1(A) = 0, \tag{2a}$$
$$\pi_3(B) = \mathbb{Z}, \tag{2b}$$
$$\pi_4(C) = 0, \tag{2c}$$
$$\mathrm{H}_1(D) = \mathbb{Z}_2, \tag{2d}$$
$$\mathrm{H}_6(E) = \mathbb{Z}. \tag{2e}$$

Which of the 5 distinct spaces in (1) corresponds to each of A, B, C, D and E?

Problem 4 (Elasticity): As we discussed in Lecture 8, in the classical theory of elasticity, the strain tensor s_{ij} and stress tensor τ_{ij} are each two-index symmetric tensors, linearly related by a rank-4 elasticity tensor:

$$\tau_{ij} = \lambda_{ijkl} s_{kl}. \tag{3}$$

Note that

$$\lambda_{ijkl} = \lambda_{klij}. \tag{4}$$

In 3 dimensions, it is often helpful to think of λ as a 6×6 matrix, with $ij = ji$ and $kl = lk$ indices spanning the 6-dimensional vector space of all linearly independent symmetric rank-2 tensors.

10 **4A:** Consider a theory which is rotation invariant, and has rotation symmetry group SO(3). Let R correspond to the representation of SO(3) corresponding to symmetric rank-2 tensors.

4A.1. Is R reducible or irreducible? If R is reducible, decompose it into irreps of SO(3).

4A.2. The elastic moduli of a theory are the eigenvalues of the 6×6 matrix λ. How many distinct elastic moduli are permitted by symmetry in this theory?

15 **4B:** Now, consider a crystal which is not isotropic. Instead, its rotational symmetry group is generated by the following transformations on the (x, y, z) coordinates:

$$a \cdot (x, y, z) \to (x, -y, -z), \tag{5a}$$
$$b \cdot (x, y, z) \to (-y, x, z). \tag{5b}$$

Let V denote this 3-dimensional "vector" representation of our symmetry group.

4B.1. What is the group G (isomorphic to)? The answer is given by one of the elementary groups described in this class, which you should explicitly give.[2]

4B.2. How does the representation V decompose into irreps of G? The needed representation theory has been covered in lecture, or in Zee, or on homework, so you should not need to re-derive it!

4B.3. How does the representation $V \otimes V$ decompose into irrreps of G?

4B.4. Explain why $V \otimes V = R \oplus V$.

4B.5. How does the representation R decompose into irreps of G?

4B.6. How many distinct elastic moduli will this crystal generically have?

[2] *Hint:* When does $a^n = 1$? What about $b^n = 1$? What does aba^{-1} equal?

20 **Problem 5:** Consider a theory in three spatial dimensions, whose order parameter consists of a pair of three-component vectors ($\mathbf{U}, \mathbf{V} \in \mathbb{R}^3$). The free energy of the system in a spatially homogenous set-up is

$$F = a|\mathbf{U}|^2 + b|\mathbf{V}|^2 + c|\mathbf{U}|^4 + d|\mathbf{V}|^4 + e(\mathbf{U} \cdot \mathbf{V})^2. \tag{6}$$

The constants $c, d, e \geq 0$, while $a, b < 0$.

5.1. Show that the minima of F have $\mathbf{U}, \mathbf{V} \neq 0$. What more can you say about $\mathbf{U}$ and $\mathbf{V}$ in the minima of F?

5.2. Explain why the order parameter space for this theory is topologically equivalent to SO(3).[3]

5.3. Now let's consider a pair of vector fields $(\mathbf{U}(\mathbf{x}), \mathbf{V}(\mathbf{x}))$. At each point $\mathbf{x}$ in some three dimensional spatial domain, we would like to minimize F given above. Would there exist any configurations where there are topological line like defects (i.e. where you can't smoothly deform your configuration to make it minimize F everywhere in space? If so, state how many different kinds of defects.

5.4. Would any point defects exist in three dimensions? If so, how many different kinds?

Problem 6 (The orbifold limit of K3): The K3 surface(s) plays an important role in string theory. It is a special class of 4-dimensional manifold onto which one can "nicely" compactify four of the extra dimensions of string theory (for reasons well beyond the scope of this class!).

One way to build a K3 surface is to perform an operation called an orbifold on a 4-dimensional torus T^4, and then "smooth out" the resulting singular space.

10 **6A:** As a warm-up exercise, let's consider the orbifold $\mathrm{S}^1/\mathbb{Z}_2$. If we think of $\mathrm{S}^1 = \mathbb{R}/\mathbb{Z}$, which corresponds to taking $x \in \mathbb{R}$ and identifying it with $x \sim x+1$, then the $\mathbb{Z}_2$ quotient in this context corresponds to additionally identifying $x \sim -x$. Because $0 \sim -0$ is mapped to the same point, we call the resulting quotient of a manifold by this $\mathbb{Z}_2$ action an **orbifold**.

6A.1. Show that the orbifold $\mathrm{S}^1/\mathbb{Z}_2$ is the line segment $[0, \frac{1}{2}]$.

6A.2. Explain why $\mathrm{S}^1/\mathbb{Z}_2$ is not a manifold.[4]

10 **6B:** Now, let's consider the orbifold $\mathrm{T}^2/\mathbb{Z}_2$. Here the $\mathbb{Z}_2$ group action takes

$$(x_1, x_2) \sim (-x_1, -x_2). \tag{7}$$

Now, both coordinates flip together! The resulting space is *not* homeomorphic to $[0, \frac{1}{2}]^2$ – so what is it?

6B.1. Show that given a point $(x_1, x_2) \in \mathbb{R}^2$, applying the equivalences (7) and $x_{1,2} \sim x_{1,2} + 1$ (the defining relation of the torus), you can identify every point in the plane with a point obeying $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq \frac{1}{2}$.

6B.2. Put a CW (or simplicial) complex structure on $\mathrm{T}^2/\mathbb{Z}_2$.[5]

6B.3. Show that the homology groups of this space are

$$\mathrm{H}_0(\mathrm{T}^2/\mathbb{Z}_2) = \mathbb{Z}, \tag{8a}$$
$$\mathrm{H}_1(\mathrm{T}^2/\mathbb{Z}_2) = 0, \tag{8b}$$
$$\mathrm{H}_2(\mathrm{T}^2/\mathbb{Z}_2) = \mathbb{Z}. \tag{8c}$$

[3] *Hint:* Can you create a set of three orthonormal vectors (that span $\mathbb{R}^3$) out of $\mathbf{U}$ and/or $\mathbf{V}$?

[4] *Hint:* What happens at the edges?

[5] *Hint:* Construct a CW complex as follows. Put 0-cells at *points* in the domain $[0, 1] \times [0, \frac{1}{2}]$ that are invariant under $\mathbb{Z}_2$. Put 1-cells on *lines* that are invariant. Where do you put 2-cells? You should find 2 2-cells, 4 1-cells and 4 0-cells.

5 **6C:** There is another perspective on (8) arising from de Rham cohomology theory. We can say that $H^2(T^2/\mathbb{Z}_2)$ should correspond to *subgroups* of $H^2(T^2)$ which are invariant under the equivalence relation (7).

6C.1. What are the cohomology groups of T^2, and the corresponding closed but not exact forms corresponding to each non-trivial element?

6C.2. Directly apply the coordinate transformation (7) to each of these forms. Which ones are invariant?

6C.3. Deduce what the cohomology groups of $T^2/\mathbb{Z}_2$ must be, and check that your answer is consistent with (8).

5 **6D:** At long last, we can now turn to K3 surfaces, which correspond to a smoothed out version of the orbifold $T^4/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ quotient corresponds to equivalence relation

$$(x_1, x_2, x_3, x_4) \sim (-x_1, -x_2, -x_3, -x_4). \tag{9}$$

6D.1. Generalize the above argument to find the number of non-trivial closed but not exact forms on $T^4/\mathbb{Z}_2$.

6D.2. It turns out that as part of the smoothing process to get from the orbifold to the manifold K3, one must *locally* remove the singular region surrounding any fixed point of the $\mathbb{Z}_2$ action, and replace it with a smoother space, at the cost of creating a single new non-trivial 2-dimensional cycle surrounding the removed point. How will this operation modify the cohomology groups?

6D.3. Conclude that the Betti numbers of K3 are $b_0 = b_4 = 1$, $b_1 = b_3 = 0$, $b_2 = 22$.