PHYS 5040: Algebra and Topology in Physics **Spring 2021**

Homework 1

- **Due:** 11:59 PM, January 25. Submit electronically on Canvas.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

15 **Problem 1 (Concatenating strings):** Consider the set consisting of the 26 Latin letters, together with the space character and the "empty character" $\emptyset$: $C = \{a, b, \ldots, z, [\text{space}], \emptyset\}$. We can build words and sentences by concatenating these elements together: for example, $t \times h \times e = the$. Also note that for any string X, $X \times \emptyset = \emptyset \times X = X$.

i. Show that C, together with the concatenation operation, does not generate a group.

ii. Consider adding the [delete] element, which behaves as follows: for any $X, Y \in C$, $[\text{delete}] \times X \times Y = Y$. Does $C \cup \{[\text{delete}]\}$, with the concatenation operations described above, generate a group? Assume $[\text{delete}] \times \emptyset = \emptyset$.

15 **Problem 2:** Consider the permutation group S_n.

i. Show that for any $i, j, k \in \{1, \ldots, n\}$,

$$(ij)(ik)(ij) = (jk) \tag{1}$$

ii. Show that for any subset of $m+1$ numbers, denoted i_m, $(i_m i_{m+1})(i_1 i_2 \ldots i_m) = (i_1 i_2 \ldots i_{m-1} i_{m+1} i_m)$.

iii. Show that all permutations in S_n can be expressed as products of the permutations $(12), \ldots, (1n)$.

15 **Problem 3:** Consider the groups $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ under addition, and the groups $\mathbb{R}^\times$ and $\mathbb{R}^\times_+ = \{x \in \mathbb{R} : x > 0\}$ under multiplication. Show which of these groups (if any) are isomorphic to each other.

Problem 4 (Reference frames): In physics, we believe that fundamental equations of the universe are the same to all observers.

10 (a) Define the Galilean symmetry group G to contain the following transformations:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}. \tag{2}$$

i. Show that the Galilean transformations (the matrices above) form a group.

ii. As we will see, it can be useful to write group elements of these continuous (Lie) groups as matrix exponentials. Show that

$$\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \mathrm{e}^{vK}, \qquad K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3}$$

We say that K generates the group.

10 (b) Next, consider the Lorentz transformations. In units where $c = 1$,

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}. \tag{4}$$

To avoid relativistic velocity addition, let's reverse the logic from part (a).

i. Expand the Lorentz transformations to first order in small v to find the generator matrix K'.
ii. Calculate $\mathrm{e}^{\theta K'}$ to find the form of a general Lorentz transformation:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}. \tag{5}$$

iii. Conclude that the Lorentz transformations form a group L.
iv. Show that (4) and (5) are equivalent (i.e. relate θ and v).

5 (c) Explain why G and L are each isomorphic to $\mathbb{R}$.

Problem 5 (Quaternions): The quaternions are a generalization of complex numbers. Define

$$\mathbb{H} = \{a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} : a, b, c, d \in \mathbb{R}\}, \tag{6}$$

where the parameters $\mathrm{i}, \mathrm{j}, \mathrm{k}$ generalize the complex number i:

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1. \tag{7}$$

Multiplication in $\mathbb{H}$ obeys the associative $[(xy)z = x(yz)]$ and distributive $[(x+y)z = xz+yz]$ properties.

10 (a) One way to think of $\mathbb{H}$ is a complex 2×2 matrix:

$$a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} = \begin{pmatrix} a + \mathrm{i}b & \mathrm{i}d + c \\ \mathrm{i}d - c & a - \mathrm{i}b \end{pmatrix}. \tag{8}$$

i. Show that (7) holds under the mapping (8).[1]
ii. Explain why quaternion multiplication can be replaced with the 2×2 matrix multiplication above. You do not (yet) need to be very rigorous.

10 (b) For $z \in \mathbb{H}$, define

$$N(z) = a^2 + b^2 + c^2 + d^2. \tag{9}$$

i. Show that $N(z)$ is equivalent to the matrix determinant in the equivalence (8).
ii. Argue that if we remove the zero quaternion 0, then $\mathbb{H} - \{0\}$ is a group under multiplication. Let's denote it $\mathbb{H}^\times$.
iii. Show that $N : \mathbb{H}^\times \to \mathbb{R}^\times$ is a homomorphism.

10 (c) Consider the set $M = \{z \in \mathbb{H} : N(z) = 1\}$.

i. Show that M is a group under quaternion multiplication.
ii. Show that M is isomorphic to $\mathrm{SU}(2)$. You may want to use (8).

[1] You should quote without explicit derivation well known Pauli matrix multiplication rules. These in turn are probably worth memorizing!

10 (d) Unit quaternions are often used to implement rotations in computer graphics due to the improved stability of quaternion multiplication over matrix multiplication. Parameterize a unit quaternion by

$$z = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\left(u_x\mathrm{i} + u_y\mathrm{j} + u_z\mathrm{k}\right). \tag{10}$$

Parameterize a vector $\mathbf{v} = (v_x, v_y, v_z)$ as a quaternion by

$$v = v_x\mathrm{i} + v_y\mathrm{j} + v_z\mathrm{k}. \tag{11}$$

i. Show that zvz^{-1} corresponds to rotation by angle α through an axis that is oriented in the direction of the unit vector $\mathbf{u} = (u_x, u_y, u_z)$. For simplicity, take $(u_x, u_y, u_z) = (1, 0, 0)$.

ii. Argue that $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$: namely, one needs to rotate through angle 4π in $\mathrm{SU}(2)$ to return to the same state.