PHYS 5040: Algebra and Topology in Physics **Spring 2021**

Homework 10

- **Due:** 11:59 PM, April 5. Submit electronically on Canvas.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

25 **Problem 1 (Quantum particle on a circle):** Consider a one dimensional quantum particle on S^1, parameterized by angle $\theta \sim \theta + 2\pi$, with Hamiltonian

$$H = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2}. \tag{1}$$

For simplicity in this problem, set $\hbar = 2I = 1$.

1.1. The eigenfunctions of H are $\psi_n(\theta) = \mathrm{e}^{\mathrm{i}n\theta}$, for $n \in \mathbb{Z}$. What are the corresponding eigenvalues?

1.2. Suppose that I have an (unphysical) initial condition $\psi(\theta, t = 0) = \delta(\theta)$.[1] Solve the time-independent Schrödinger equation to find $\psi(\theta, t)$. The answer will be expressed as an infinite sum.

1.3. Use the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}n\theta}\mathrm{e}^{-an^2} = \sqrt{\frac{\pi}{a}} \sum_{q\in\mathbb{Z}} \mathrm{e}^{-(\theta-2\pi q)^2/4a} \tag{2}$$

to re-express your earlier answer in another form.

1.4. Explain why the sum over integers q can be interpreted as a topological "winding number" for the quantum particle.[2]

25 **Problem 2 (Fundamental theorem of algebra):** In this problem, we will use homotopy theory to prove the fundamental theorem of algebra, which (for simplicity) we state as follows: if $n \geq 1$ and

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \tag{3}$$

is a complex polynomial ($a_{n-1}, \ldots, a_0, z \in \mathbb{C}$), then $p(z) = 0$ has a solution.

We will prove this theorem by contradiction. Suppose $p(z) = 0$ has no solution for any $z \in \mathbb{C}$. Then consider the function $F : \mathbb{C} \to \mathrm{S}^1$ defined as

$$F(r, \theta) = \frac{p(\mathrm{e}^{\mathrm{i}\theta} r)p(r)^{-1}}{|p(\mathrm{e}^{\mathrm{i}\theta} r)p(r)^{-1}|} \tag{4}$$

is well-defined.

[1] This initial condition should be thought of as giving us a Green's function for the Schrödinger equation.

[2] *Hint:* Think about solving the problem another way, by using $\mathrm{S}^1 = \mathbb{R}/\mathbb{Z}$ and knowing the solution to the free particle Schrödinger equation on $\mathbb{R}$.

2.1. Show that there exists a large value of ρ such that $F(\rho,\theta) : \mathrm{S}^1 \to \mathrm{S}^1$ has winding number n.

2.2. By considering $F(t\rho,\theta)$ for $0 \le t \le 1$, also show that $F(\rho,\theta)$ has winding number 0. Conclude the fundamental theorem of algebra.

30 **Problem 3 (Linking number of DNA):** DNA is a double-helical polymer. There is a topological number called the linking number L, associated with the number of times that the DNA strands wind around each other.

For simplicity, in this problem we will consider closed loops of DNA, which are common in bacteria (where a single loop of DNA floats around inside of the cell). If $\mathbf{r}_{1,2}(s)$ are a pair of (closed) curves in space, corresponding to the locations of the DNA strands, with $\mathbf{r}_{1,2}(0) = \mathbf{r}_{1,2}(1)$, then

$$L = \frac{1}{4\pi}\int \mathrm{d}s_1 \int \mathrm{d}s_2 \left(\frac{\mathrm{d}\mathbf{r}_1}{\mathrm{d}s} \times \frac{\mathrm{d}\mathbf{r}_2}{\mathrm{d}s}\right) \cdot \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \tag{5}$$

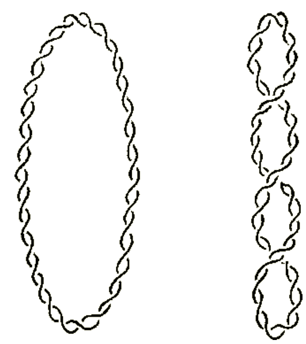

Figure 1: A sketch of how DNA is generically found in nature, both in its energetically minimal form (left) and in a strained form (right) that allows for the local unwinding of the helix. From Wikipedia.

To get some intuition for why L is a topological number, consider the simple scenario where we can approximate

$$\mathbf{r}_1(s_1) \approx \rho(s_1)\cos\phi(s_1)\hat{\mathbf{x}} + \rho(s_1)\sin\phi(s_1)\hat{\mathbf{y}} + z(s_1)\hat{\mathbf{z}} \tag{6a}$$

$$\mathbf{r}_2(s_2) \approx s_2\hat{\mathbf{z}}. \tag{6b}$$

Treat the domain $-\infty < s < \infty$ when evaluating the integrals (Imagine that for $|s|$ very large, $\mathbf{r}_2$ and ultimately $\mathbf{r}_1$ as well are joined back up without wrapping around each other anymore).

3.1. Do the s_2 integral upon plugging in (6) into (5), and show that

$$L = \frac{1}{2\pi}\int \mathrm{d}s_1 \frac{\mathrm{d}\phi}{\mathrm{d}s_1}. \tag{7}$$

3.2. Conclude that L must be an integer.

3.3. To understand the topological origin of this integer, let's consider the domain where strand 1 can exist:

$$\mathcal{R} := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 > 0\}. \tag{8}$$

Construct an "explicit" deformation retraction of $\mathcal{R}$ onto S^1, either with equations or pictures.

3.4. What is $\pi_1(\mathcal{R})$?

3.5. Relate L to the fundamental group.

In nature, one expects that (very roughly) $L \approx 0.09 \times (\text{base pairs})$ – the winding double helix of DNA is very topologically non-trivial! But one also often finds that *globally*, the DNA molecule winds around itself some number of times in the opposite direction, reducing L a little further! A possible advantage to this is that this allows for local "unwinding" of the DNA double helix, which would make it easier for DNA to interact with other proteins (such as the polymerase that spits out the RNA strands which in turn lead to protein creation).

30 **Problem 4 (Energy of a superfluid vortex):** In this problem, we will consider the effective theory of superfluidity. As discussed in Lecture 17, this is captured by an angular degree of freedom $\theta \in \mathrm{S}^1$. Let us consider this theory in the annular domain

$$R = \{(r,\phi) : \phi \sim \phi + 2\pi,\ a < r < L\}. \tag{9}$$

Here ϕ is an angular coordinate; the radius r extends from the small scale a (the size of the superfluid vortex, where our effective theory breaks down), to the system size L of the box (or, the distance to another vortex). The energy of the superfluid is given by

$$E[\theta] = \int_a^L r\mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi\, \frac{\hbar^2\psi_0^2}{2m}\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}r}\right)^2 + \frac{1}{r^2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2\right]. \tag{10}$$

4.1. Consider the solution

$$\theta(r,\phi) = n\phi + \theta_0(r,\phi), \tag{11}$$

where $\theta_0(r,\phi)$ is a single-valued function. Use homotopy theory to explain why this is the most general possible configuration $\theta : R \to \mathrm{S}^1$.[3]

4.2. Show that

$$E[\theta] = E_n + E[\theta_0], \tag{12}$$

where

$$E_n = \frac{\pi\hbar^2\psi_0^2}{m}\log\frac{L}{a} \times n^2 \tag{13}$$

can be interpreted as the minimal energy of the vortex configuration with "vortex charge" n. Hence, topological superfluid configurations are intrinsically higher energy – but this energy cannot easily be dissipated away once it is created!

4.3. Estimate the *free energy* $F = E - TS$ of a vortex of winding number $n = 1$, in a box of size L. You can estimate $E \approx E_1$ (even if the box is not annular shaped!); the entropy comes from all the possible positions in a two dimensional box of size $\sim L$ that the vortex might occupy. Show that there is a critical temperature T_c above which a vortex has negative free energy, and estimate an expression for T_c. Does it depend on L?

4.4. Suppose that we have a superfluid in a disk shaped region D. Even when $T > T_\mathrm{c}$, we observe that

$$\int_{\partial R} \mathrm{d}\theta = 0. \tag{14}$$

Why must this happen?

4.5. Reconcile your answers to the previous two parts, based on physical intuition. What do you think will happen to the superfluid as we heat T above T_c?[4]

The phase transition that occurs at T_c is called the Kosterlitz-Thouless transition.

[3] *Hint:* Why can n not depend on r?

[4] *Hint:* If we create M vortices of the same absolute $|n|$, we might approximate that their combined free energy is $F_n \times M$.