Homework 11

- **Due:** 11:59 PM, April 12. Submit electronically on Canvas.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

20 **Problem 1 (Lens spaces):** Consider a sphere S^{2k-1}, with $k \geq 1$ an integer. We can think of this sphere as a subspace of the complex plane $\mathbb{C}^k$:

$$\mathrm{S}^{2k+1} = \{(z_1, \dots, z_k) \in \mathbb{C}^k : |z_1|^2 + \cdots + |z_k|^2 = 1\}. \tag{1}$$

Let p be a positive integer. Consider the following equivalence relation on C^k:

$$(z_1, \dots, z_k) \sim (\mathrm{e}^{2\pi\mathrm{i}/p} z_1, \dots, \mathrm{e}^{2\pi\mathrm{i}/p} z_k). \tag{2}$$

1.1. Show that this equivalence relation is well-defined on S^{2k-1} as well.

1.2. Show that each point on S^{2k-1} is identified with $p-1$ other points by $\sim$.

1.3. Define $\mathrm{L}(2k-1, p) := \mathrm{S}^{2k-1}/\sim$. What is $\pi_1(\mathrm{L}(2k-1,p))$?

$\mathrm{L}(2k-1, p)$ is called a **lens space**. These spaces played rather important roles in the history of topology, as it turns out one can find a variety of lens spaces which are not homeomorphic, but which have all of the same homotopy groups. Hence, the homotopy classification of topological spaces is incomplete!

20 **Problem 2:** Consider the map $\gamma : \mathrm{S}^1 \to \mathrm{SO}(3)$ given by

$$\gamma(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Here $\theta \sim \theta + 2\pi$ corresponds to the angular coordinate on the circle S^1.

2.1. It turns out that $[\gamma]$ is the non-trivial element of $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$. Based on this fact, use the relationship between SO(3) and SU(2) to explain how to lift this into a non-closed path in SU(2).

2.2. If we traverse γ twice in SO(3), the resulting loop should lift to a loop in SU(2). For this lift into SU(2), construct an explicit homotopy:

$$\mathit{\Gamma}(\theta, t) = a(\theta, t) \cdot 1 + b(\theta, t) \cdot \mathrm{i}\sigma^x + c(\theta, t) \cdot \mathrm{i}\sigma^z, \tag{4}$$

which goes from the "doubled γ" loop to a point in SU(2).

2.3. Now, convert this homotopy into a homotopy in SO(3).[1]

[1] *Hint:* If $\mathit{\Gamma}(\theta, t) = \cos\phi + \mathrm{i}\sin\phi n_j \sigma^j$, then it corresponds to a rotation by angle 2ϕ around an axis oriented along the unit vector n_j. If you know the generators T_j of the Lie algebra of SO(3) (or SU(2)) in the spin-1 representation, then how can you construct $\mathit{\Gamma}(\theta, t)$ in SO(3)? You don't need to carry out all of the algebra, which gets messy.

20 **Problem 3:** Consider the free group $G = \mathbb{Z} * \mathbb{Z}$, which is generated by two letters (e.g. a and b). Find a subgroup of G which is isomorphic to $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ (namely, a free group on 3 generators).[2]

20 **Problem 4:** Consider a regular two-dimensional polygon with n sides. Let P_n be the set of all points which either lie inside, or on the boundary, of the polygon.

4.1. What is $\pi_1(X)$?

4.2. Show how to identify points in P_n with one another (via an equivalence relation $\sim$, which you should construct explicitly) to create a topological space $X = P_n/\sim$ with $\pi_1(X) = \mathbb{Z}_n$.

Problem 5 (Biaxial nematic liquid crystals): In this problem, we will study a more exotic (and less common) type of liquid crystal, whose order parameter space X can be thought of as the space of all orthogonal lines passing through the origin. Alternatively, we can think of X as classified by an (ordered) orthogonal triplet of unit vectors in $\mathbb{R}^3$, up to sign:

$$X = \{(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) : \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}\}/[\mathbf{e}_i \sim -\mathbf{e}_i, \text{for each } i \text{ separately}]. \tag{5}$$

The ordering of unit vectors is important, and cannot be swapped under the equivalence relation.

20 **5A:** Let's begin by calculating $\pi_1(X)$.

5A.1. Argue that you can think of $X = \mathrm{SO}(3)/\sim$, where (with a slight abuse of notation) $\sim$ corresponds to the equivalence relation where exactly two of the three $\mathbf{e}_i$ can be flipped.

5A.2. Using this result, explain why the universal covering space of X must be SU(2).

5A.3. Next, argue that the fundamental group $\pi_1(X) = Q$, where

$$Q = \{1, -1, \mathrm{i}\sigma^x, -\mathrm{i}\sigma^x, \mathrm{i}\sigma^y, -\mathrm{i}\sigma^y, \mathrm{i}\sigma^z, -\mathrm{i}\sigma^z\} \tag{6}$$

is an 8 element subgroup of SU(2), often called the discrete quaternion group.[3]

20 **5B:** Now, let's use homotopy theory to classify two dimensional topological defects in biaxial nematics, and understand how they interact with each other.

5B.1. Draw a "defect" corresponding to the element "1" of Q. More precisely, sketch the simplest example of how the order parameter changes as you wind around a loop encircling the point defect.

5B.2. Draw a "defect" which could correspond to the element "$\mathrm{i}\sigma^x$" of Q. [4]

5B.3. What kind of defect do you get if you combine 4 of the "$\mathrm{i}\sigma^x$" defects? As best you can, sketch a picture explaining your answer (though it may be hard to draw and visualize!).

5B.4. What happens if we try to move a "$\mathrm{i}\sigma^y$" defect around an "$\mathrm{i}\sigma^x$" defect?

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[2] *Hint:* Define c, d and e as appropriate products of a and b, such that there is no non-trivial string $cdcccec^{-1}d\cdots$ (e.g.) that can equal the identity.

[3] *Hint:* What complex matrix in SU(2) corresponds to a 180° rotation about an axis? Why is this important?

[4] *Hint:* For both this question and the next, it may be helpful to begin by thinking about the non-trivial defect in the ordinary nematic liquid crystal. Why do they become more complicated in the biaxial case?