PHYS 5040: Algebra and Topology in Physics — Spring 2021

Homework 12

- **Due:** 11:59 PM, April 19. Submit electronically on Canvas.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

20 **Problem 1:** The space $\mathbb{R}\mathrm{P}^n$ is defined as

$$\mathbb{R}\mathrm{P}^n := (\mathbb{R}^{n+1} - \{\mathbf{0}\})/[\mathbf{x} \sim \lambda \mathbf{x},\ (\lambda \neq 0)]. \tag{1}$$

It can be thought of as the set of all lines passing through the origin. Assume that $n \geq 2$.

1.1. Explain why $\mathbb{R}\mathrm{P}^n = \mathrm{S}^n/\mathbb{Z}_2$.

1.2. Calculate $\pi_n(\mathbb{R}\mathrm{P}^n)$.

25 **Problem 2 (Solitons):** Consider a theory with order parameter S^1, living on the real line $\mathbb{R}$. The theory can be understood by considering maps $\theta(x) : \mathbb{R} \to \mathrm{S}^1$ such that $\theta(x) \sim \theta(x) + 2\pi$ are equivalent. Now suppose we add a pinning field h, which tries to pin the field to $\theta = 0$. If there is some "bending energy" associated with the order parameter fluctuating in space, we may write the energy of a configuration as

$$E = \int \mathrm{d}x \left[\frac{K}{2} \left(\frac{\mathrm{d}\theta}{\mathrm{d}x} \right)^2 - h \cos\theta \right]. \tag{2}$$

Clearly, the lowest energy configuration is to just have $\theta(x) = 0$. However, there are "stable" energetic configurations called **solitons**. In a soliton configuration, the $\theta(x)$ configuration corresponds to a function which tends to $\theta = 0$ both as $x \to \pm\infty$, but winds around S^1 one time in the middle of the line.

2.1. Explain why soliton configurations are classified by $\pi_1(\mathrm{S}^1) = \mathbb{Z}$.

2.2. Describe how to find the profile $\theta(x)$ corresponding to a soliton configuration of *minimal energy* by solving the Euler-Lagrange equations to minimize the functional above.[1]

2.3. Suppose that I give you the following configuration:

$$\theta(x) = \begin{cases} 0 & x < 0 \\ x/a & 0 \leq x \leq 4\pi a \\ 4\pi & x > 4\pi a \end{cases} \tag{3}$$

Here a is a length scale which just fixes units (but otherwise doesn't matter for this problem). *Qualitatively* estimate what will happen to the $\theta(x)$ configuration at very long times, assuming that the (presumably dissipative) time dynamics we associate to the $\theta(x)$ field tends to minimize the total energy E at late times.[2]

[1] *Hint:* Make an analogy to Lagrangian mechanics. In this analogy, you are studying a mechanical problem in one dimension, so "energy conservation" in the mechanical problem allows you to replace Newton's Laws with a *first order* differential equation. Solve this simpler equation to find $\theta(x)$ as the solution to an integral equation.

[2] *Hint:* Try to generalize the calculation of 2.2 to general winding number n. Can you? If not, what does it physically imply?

Problem 3 (Skyrmions): A skyrmion is a topologically protected object; one example of which can arise in condensed matter physics via models of two-dimensional thin-film magnetism. Let $\mathbf{M}(\mathbf{x}) : \mathbb{R}^2 \to \mathbb{R}^3$ denote a two-dimensional map to three-dimensional vectors corresponding to the local magnetization in the sample.

20 **3A:** Consider the phenomenological free energy

$$F = \int \mathrm{d}^2\mathbf{x} \left[A(\partial_x \mathbf{M})^2 + A(\partial_y \mathbf{M})^2 - B\mathbf{M}^2 + C(\mathbf{M}^2)^2 \right] \tag{4}$$

where A, B and C are all positive constants.

3A.1. Assume that the magnet is homogeneous, so that $\mathbf{M}$ does not depend on $\mathbf{x}$. What are the minima of the free energy?

3A.2. What topological space can we associate with the order parameter $\mathbf{M}$, governing this ordered phase of matter?

3A.3. What is the symmetry group of this magnet? Namely, for what group G is there a natural group action on the order parameter $\mathbf{M}$, which leaves F invariant: $F[\mathbf{M}] = F[g \cdot \mathbf{M}]$?

20 **3B:** A skyrmion defect corresponds to the following choice of magnetic field profile $\mathbf{M}$:

$$\mathbf{M}(\mathbf{x}) = \alpha(\sin\theta(r)\cos(n\phi), \sin\theta(r)\sin(n\phi), \cos\theta(r)). \tag{5}$$

Here r and ϕ denote polar coordinates in the plane $\mathbb{R}^2$; the precise function $\theta(r)$ is not relevant for this calculation, beyond its asymptotics: $\theta(\infty) = 0$ while $\theta(0) = \pi$.

3B.1. What value of α will minimize the free energy (4)?

3B.2. Skyrmion defects are classified by a certain homotopy group of a certain topological space. What is the relevant space and what is the relevant homotopy group? (State both which π_n it is, and what the actual group is for this particular space.)

3B.3. How does this topological classification constrain, and/or appear, in the ansatz (5)?

3B.4. Consider the integral

$$I = \frac{1}{4\pi} \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{x}\, \mathbf{M} \cdot \partial_x \mathbf{M} \times \partial_y \mathbf{M} = \frac{1}{4\pi} \int_{\mathbb{R}^2} \mathrm{d}r\mathrm{d}\phi\, \mathbf{M} \cdot \partial_r \mathbf{M} \times \partial_\phi \mathbf{M}. \tag{6}$$

The spatial integral runs over the whole plane. Show that on the ansatz (5), this integral captures precisely the homotopy class of this skyrmion defect.

25 **Problem 4:** Some liquid crystal molecules are shaped like flat disks. Assume that the flat disk looks the same both from above and below.

4.1. What is the order parameter space for this system?[3]

4.2. What kinds of topological defects can exist in such disk-shaped liquid crystals in three dimensions? (Make sure to consider all possibilities.)

[3] *Hint:* This is a space we have seen before. The disks live in three dimensional space.