PHYS 5040: Algebra and Topology in Physics **Spring 2021**

Homework 13

- **Due:** 11:59 PM, April 26. Submit electronically on Canvas.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

Problem 1 (Finite element methods and electromagnetism): Consider a three dimensional space built out of a mesh of vertices (in set V), oriented edges (in set E), oriented faces (in set F) and oriented "cubes" (in set C) (or more generally, filled in three-dimensional regions). How might we naturally solve the equations of electromagnetism on such a grid? Remarkably, the answer is closely related to much of the mathematics we've been developing.

Let us formally define a vector space $\mathbb{R}^V$, made up out of elements of the form

$$|\phi\rangle = \sum_{v\in V} \phi_v |v\rangle, \qquad \langle v|v'\rangle = \delta_{vv'}. \tag{1}$$

In particular, if we wanted to specify the voltage ϕ in our domain, we would specify the voltage at each discrete point $v \in V$, $\phi_v = \phi(v)$. This data is conveniently collected into the finite-dimensional numerical vector $|\phi\rangle$, a numerical interpolation of the continuous voltage function in a physical three dimensional space.

We can analogously define $\mathbb{R}^E$, $\mathbb{R}^F$ and $\mathbb{R}^C$ for edges, faces and cubes. Given a 1-form ω, for example, we can numerically approximate it via

$$\omega = \sum_{e\in E} \omega_e |e\rangle, \qquad \langle e|e'\rangle = \delta_{ee'}. \tag{2}$$

Here the coefficient is meant to numerically capture

$$\omega_e \approx \int_e \omega. \tag{3}$$

15 **1A:** We can use what we know about homology and cohomology to induce natural notions of derivatives on this discrete mesh. Indeed, in the exact sequence used to define the homology groups, we found boundary homomorphisms $\partial_{1,2,3}$ between various chain groups, and they naturally find analogues here too. For example, if edge $e = (uv)$ is an oriented edge from vertex u to vertex v, we might define a matrix $D_1 : \mathbb{R}^E \to \mathbb{R}^V$ such that

$$D_1|e\rangle = |\partial e\rangle = |v\rangle - |u\rangle. \tag{4}$$

D_2 and D_3 are defined analogously by generalizing the boundary operator from faces to edges, and cubes to faces.

1A.1. What is the matrix $D_1 D_2$?

1A.2. Suppose that we have a 1-chain $\sigma = e_1 + e_2 - e_3 + \cdots$. Show that if we define

$$|\sigma\rangle = |e_1\rangle + |e_2\rangle - |e_3\rangle + \cdots, \tag{5}$$

then given our numerical approximation to the 1-form ω in (3),

$$\int_\sigma \omega = \langle \sigma | \omega \rangle. \tag{6}$$

Explain how this generalizes to other p-forms.

1A.3. We want to find an operator which corresponds to our numerical approximation to the exterior derivative d. We can define it by demanding that given a p-form χ, that the numerical approximation to the $(p+1)$-form $\mathrm{d}\chi$ automatically obeys Stoke's Theorem:

$$\langle \sigma | \mathrm{d}\chi \rangle = \langle \partial\sigma | \chi \rangle. \tag{7}$$

Justify (7), and use it to determine the matrix $\mathcal{D}_p$ which takes a p-form χ and generates $|\mathrm{d}\chi\rangle = \mathcal{D}_p|\chi\rangle$.

1B: Now that we have our derivative operator in hand, let's determine how we would solve Maxwell's equations. For simplicity, let us work in an unusual (to a physicist) gauge where we fix the voltage $\phi = 0$ everywhere. In this case, the electromagnetic fields are determined by finding a 1-form field A. The magnetic field $B = \mathrm{d}A$ is a 2-form, and the electric field $E = -\partial_t A$ is a 1-form.

1B.1. Write Faraday's Law in terms of differential forms. Show that our gauge choice is compatible with this equation.

1B.2. Evaluate $\mathrm{d}B$.

1B.3. Using Ohm's Law, we can define a current 1-form $J = \sigma E$, where σ is a constant called the conductivity. Show that Ampére's Law can be written as

$$\mu\sigma E = *\mathrm{d} * B \tag{8}$$

where μ is the permeability of the material. Treat μ and σ as constants in what follows.

1B.4. Explain why the following numerical prescription is sensible: for 3-chain R and p-form χ,

$$\int_R \chi \wedge *\chi = \langle \chi | \mathcal{V}_p | \chi \rangle, \tag{9}$$

where the matrix $\mathcal{V}_p$ is diagonal (do not worry about a precise determination of this factor).[1]

1B.5. Apply $\cdots \wedge *f$ to both sides of (8); here f is an arbitrary 1-form called a "test function". By using the identities developed thus far, show how to obtain a set of *ordinary differential equations* for a time-dependent vector $|A(t)\rangle \in \mathbb{R}^E$ which corresponds to the vector potential, and whose solutions approximately solve (8) (up to finite-size effects, etc.).[2]

1C: Let us imagine solving Maxwell's equations in a spatial domain T, which is a thin (metallic) shell which is warped into the shape of a donut's surface.

[1] *Hint:* If, e.g., $p = 1$, can you show that $\chi \wedge *\chi$ is a volume 3-form multiplying $\chi_i\chi_i$?
[2] *Hint:* If for all $|f\rangle$, $\langle f|a\rangle = \langle f|b\rangle$, then $|a\rangle = |b\rangle$.

1C.1. Find the cohomology groups of this space.[3]

1C.2. How many possible electric field configurations E exist which cannot be obtained by any well-defined voltage function? How does the answer relate to cohomology?

1C.3. For each such configuration, explain physically how it might be created. In order to do so, you may feel free to imagine placing little wire loops inside or outside the donut, and run a current through them.

1C.4. Show that there is (at least one) closed but not exact 2-form B in domain T. Draw physically what such a magnetic field configuration would correspond to.

1C.5. We might then conclude that to keep track of the physical magnetic field, we need to keep track of A *and* some additional degree of freedom B_{top}, for the "topological" contribution to the magnetic field. However, show that B_{top} has no dynamics and therefore, we do not need to modify our equation of motion for A.

30 **Problem 2:** Consider an equilateral triangle T, with the three vertices at the corners labeled as a, b, c. Consider the equivalence relation which identifies $a \sim b \sim c$. Define $X = T/\sim$.

2.1. Put a CW complex structure on the space X.

2.2. Calculate $\pi_1(X)$.

2.3. Calculate $\mathrm{H}_0(X)$, $\mathrm{H}_1(X)$, and $\mathrm{H}_2(X)$.

30 **Problem 3:** Consider the complex projective plane

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} - \{0\})/[(z_1, \ldots, z_{n+1}) \sim \lambda(z_1, \ldots, z_{n+1}), \lambda \in \mathbb{C} - \{0\}]. \tag{10}$$

3.1. Consider the coordinates $(z_1, \ldots, z_{n+1})$ which (up to equivalence relation) describe $\mathbb{CP}^n$). Show that the subset of all points in $A \subset \mathbb{CP}^n$ which have $z_1 \neq 0$ is topologically equivalent to the plane $\mathbb{C}^n$.

3.2. Explain why A is topologically equivalent to a $2n$-dimensional disk; namely, a $2n$-cell in a tentative CW complex on $\mathbb{CP}^n$.

3.3. The boundary ∂A is glued to a subspace $B \subset \mathbb{CP}^n$, corresponding to all points where $z_1 = 0$. Explain why B is topologically equivalent to $\mathbb{CP}^{n-1}$.

3.4. Conclude that there is a CW complex on $\mathbb{CP}^n$ with one 0-cell, one 2-cell, etc., up to one $2n$-cell.

3.5. Find all homology groups $\mathrm{H}_k(\mathbb{CP}^n)$, for all values of k.

Problem 4 (Weyl fermions and Berry curvature): In solid-state and particle physics, Weyl fermions are a chiral particle with a "topologically protected" dispersion relation (energy as a function of momentum/wave number). To avoid relativistic quantum mechanics, let us focus our discussion on the solid-state physics setting, where we can think about a band structure in some complicated material where the effective Hamiltonian describing non-interacting (let's say) spin[4] up and spin down electrons with wave number $\mathbf{k}$ is

$$H = A(k_x\sigma^x + k_y\sigma^y + k_z\sigma^z), \tag{11}$$

where σs denote Pauli matrices, and A is an overall constant to fix the dimensions.

[3] *Hint:* Find a deformation retraction of T onto a space whose homology groups we have already calculated.

[4] In reality, this "spin" may not be the physical electron spin angular momentum. But mathematically you can think of it as the usual spin, for this problem.

20 **4A:** Let's begin by analyzing the eigenstates of H.

4A.1. Show that the eigenvalues of H (at fixed $\mathbf{k}$) are $\pm A|\mathbf{k}|$.[5]

4A.2. Let

$$k_x = k\sin\theta\cos\phi, \quad k_y = k\sin\theta\sin\phi, \quad k_z = k\cos\theta. \tag{12}$$

Show that an eigenvector of H (for fixed $|\mathbf{k}| = k$) is given by

$$|\psi_{\mathbf{k}}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \mathrm{e}^{\mathrm{i}\phi}\sin\frac{\theta}{2} \end{pmatrix}. \tag{13}$$

It's a little unnerving that $|\psi\rangle$ seems ill-defined at $\theta = \pi$. But since wave functions aren't globally defined anyway up to a phase, perhaps this is fine?

4A.3. The **Berry curvature** is a differential form in $\mathbf{k}$-space, defined as

$$A = -\mathrm{i}\langle\psi_{\mathbf{k}}|\frac{\partial|\psi_{\mathbf{k}}\rangle}{\partial\theta}\mathrm{d}\theta - \mathrm{i}\langle\psi_{\mathbf{k}}|\frac{\partial|\psi_{\mathbf{k}}\rangle}{\partial\phi}\mathrm{d}\phi. \tag{14}$$

Use (13) to evaluate the **Berry flux** $F = \mathrm{d}A$.

4A.4. Now, evaluate

$$I = \int\limits_{|\mathbf{k}|=k} F \tag{15}$$

Note that the integration surface is the sphere S^2 parameterized by our standard spherical coordinates θ and ϕ. What do you find for the value of the integral (evaluate it by converting it to an ordinary two-dimensional integral as soon as possible).

4A.5. Explain why $I \neq 0$ is inconsistent with the existence of a globally defined A.

20 **4B:** The fact that the Berry flux is non-zero is "topological", for a few different reasons that we will now explore. To begin, recall that a normalized quantum state $|\psi\rangle$ obeying $\langle\psi|\psi\rangle = 1$ is identifiable with itself up to a phase. If we relax normalization, we might say that $|\psi\rangle \sim \lambda|\psi\rangle$ where λ is any non-zero complex number.

4B.1. Explain why the wave functions $|\psi_k\rangle$ (with k fixed) correspond to a map from S^2 to S^2.[6]

4B.2. Use homotopy theory to argue that it's possible for $|\psi_{\mathbf{k}}\rangle$ to be non-trivial, in that it cannot be deformed to a constant map.

4B.3. Argue that $|\psi_{\mathbf{k}}\rangle$ above corresponds to such a non-trivial map.[7]

4B.4. A more physical argument for this non-trivial homotopy is that I must be quantized. It turns out that for any loop $\gamma \subset \mathrm{S}^2$, the **Berry phase**

$$\mathrm{e}^{\mathrm{i}\phi} = \exp\left[\mathrm{i}\int\limits_{\gamma} A\right] \tag{16}$$

must be a well-defined complex number of unit norm. By considering γ as the boundary of two distinct regions on the k-sphere (e.g., the region containing the north pole, or the region containing the south pole), show that the integral $\int_\gamma A$ is not uniquely defined as a real number. Conclude that for some constant κ (which you should find), we must have $I = \kappa n$ for $n \in \mathbb{Z}$. This n relates to the homotopy class of $|\psi_{\mathbf{k}}\rangle$.

[5] *Hint:* If you do the brute force computation, you only have to find the eigenvalues of a 2×2 matrix.

[6] *Hint:* One S^2 corrresponds to $\mathbf{k}$; so the other must correspond to $|\psi\rangle$. How?

[7] *Hint:* Can you map $|\psi\rangle$'s to points on an ordinary sphere embedded in 3-dimensional space? Think about Problem 3.