Homework 2

- **Due:** 11:59 PM, February 1. Submit electronically on Canvas.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

Problem 1 (Alternating group): There is an important homomorphism from the permutation group $\mathrm{S}_n \to \mathbb{Z}_2$, called the **sign of a permutation**. Intuitively, the idea is as follows: define a transposition to be a permutation that exchanges two elements, such as $(i_1 i_2)$. The sign of a permutation σ is even ($\mathrm{sign}(\sigma) = 1$) if σ can be written as a product of an even number of swaps, and odd ($\mathrm{sign}(\sigma) = -1$) if σ can be written as an odd number of swaps. It is not obvious at first glance, however, that this sign operation is well-defined, or that it could be a group homomorphism. (Note that the group $\{\pm 1\}$, under multiplication, is isomorphic to $\mathbb{Z}_2$).

20 (a) To help us show that this indeed is all well-defined, consider the following polynomial:

$$P = \prod_{1 \le i < j \le n} (x_i - x_j). \tag{1}$$

There is a natural group action of S_n onto the set $\{P, -P\}$:

$$\sigma \cdot P = \prod_{1 \le i < j \le n} (x_{\sigma(i)} - x_{\sigma(j)}). \tag{2}$$

We define

$$\mathrm{sign}(\sigma) = \begin{cases} 1 & \sigma \cdot P = P \\ -1 & \sigma \cdot P = -P \end{cases}. \tag{3}$$

i. Explain in one or two sentences why $\sigma \cdot P$ either returns P or $-P$.

ii. Show that "sign" is a group homomorphism.

iii. Show that for all transpositions, $\mathrm{sign}((i_1 i_2)) = -1$.

iv. Conclude that S_n has a normal subgroup consisting of all even permutations. This is called the **alternating group** A_n.

v. Which of the groups that we have seen in class is isomorphic to A_3?

5 (b) Explain, as tersely as you can, how to use the sign of a permutation to define the n-dimensional Levi-Civita tensor.[1]

[1] This object is commonly used to define cross products, etc., in Einstein index notation. If you are not familiar with it, look it up online!

20 **Problem 2 (Chiral molecules):** Consider a molecule with a central carbon atom, bonded to sub-units labeled A, B, C, D, as shown in Figure 1. Although this molecule is in general very asymmetric, we can rotate it freely in space, so we might think that, a priori, all possible arrangements of A, B, C, D are equivalent.

i. Consider the set of all rotations which keep the central carbon fixed, together with reflections through any plane passing through the carbon atom. Show how to apply these transformations in such a way as to arbitrarily reshuffle the A,B,C,D subunits amongst themselves.

ii. Argue that there is a subgroup $G \leq \mathrm{O}(3)$ isomorphic to S_4. Here you can think of the group elements of S_4 as describing how to rearrange the 4 subunits.

iii. In a few sentences, explain why the determinant evaluated in G is equivalent to the sign of a permutation in S_4 (rigor is not needed).

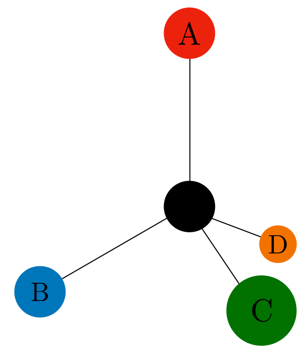

Figure 1: A chiral molecule with a central carbon bonded to 4 distinct subunits A, B, C, D.

iv. In the real world, we can only continuously rotate a molecule via $\mathrm{SO}(3)$ transformations. Argue that there are 2 kinds of configurations of subunits, which we might call "left-handed" and "right-handed", which are physically distinct. This is the origin of molecular chirality.

v. If the A, B, C and D are all individual atoms, is it possible to have a chiral molecule if any two of those atoms are identical?

Problem 3 (Euclidean group): The Euclidean group $\mathrm{E}(n)$ has elements (A_{ij}, a_i) consisting of an orthogonal matrix $A_{ij} \in \mathrm{O}(n)$ and a vector $a_i \in \mathbb{R}^n$. One way to define it is by a group action $\mathbb{R}^n$:

$$(A, a) \cdot x_i = A_{ij} x_j + a_i. \tag{4}$$

10 (a) Show that $\mathrm{E}(n)$ is a group, and that $\mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$.

15 (b) Consider a Lagrangian for a point particle of mass m and charge q moving in 2 spatial dimensions, in a magnetic field of strength B:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{qB}{2}\left(\dot{x}y - \dot{y}x\right). \tag{5}$$

i. Show that when $B = 0$, the Lagrangian is invariant under $\mathrm{E}(2)$ (acting in the obvious way on the coordinates x and y).

ii. Now consider the case $B \neq 0$. In this case, the Lagrangian is clearly not invariant under $\mathrm{E}(2)$, but what is the symmetry group of the theory? Since we only care whether the *action* is invariant, consider a symmetry to be any transformation which leaves L unchanged up to a total derivative:

$$L \to L + \frac{\mathrm{d}}{\mathrm{d}t} f(x, y, \dot{x}, \dot{y}). \tag{6}$$

Show that the magnetic field B breaks the symmetry group $\mathrm{E}(2) \to \mathbb{R}^2 \rtimes \mathrm{SO}(2)$.

iii. What happens if we try to apply one of the transformations that is explicitly not a symmetry when $B \neq 0$? Comment on the result.

Problem 4 (Crystals with cubic symmetry): In this problem, we will explore the emergent symmetries and symmetry breaking patterns that can take place inside of a two-dimensional crystal with cubic symmetry. Here, the cubic symmetry group is (isomorphic to) D_8, and acts as follows: writing

$$\mathrm{D}_8 = \left\langle r, s \,\middle|\, r^4 = s^2 = 1, rs = sr^3 \right\rangle, \tag{7}$$

we define the action of D_8 on a two dimensional vector as follows:

$$r \cdot \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a_y \\ a_x \end{pmatrix}, \tag{8a}$$

$$s \cdot \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a_x \\ a_y \end{pmatrix}. \tag{8b}$$

20 (a) Let us consider the formation of spontaneous magnetization inside of our cubic crystal. The order parameter for this phase transition is the magnetization vector (M_x, M_y).

i. The most general possible free energy for this system might be written as $F(M_x, M_y)$. What are the constraints on this function imposed by D_8 symmetry?

ii. Explain why there are two possible patterns of symmetry breaking that are qualitatively different; what are the unbroken subgroups corresponding to each?

iii. Check that the number of cosets corresponds to the number of degenerate minima for each of the 2 patterns of symmetry breaking.

20 (b) Now, let us consider the spontaneous deformation of the crystal itself, which can sometimes occur upon cooling down a solid to lower temperatures (this is called a structural phase transition). Here, the order parameter is a symmetric 2×2 matrix (the strain tensor), which we will parameterize as

$$\begin{pmatrix} s_{xx} & s_{xy} \\ s_{yx} & s_{yy} \end{pmatrix} = \begin{pmatrix} s_0 + s_+ & s_\times \\ s_\times & s_0 - s_+ \end{pmatrix}. \tag{9}$$

Consider s_+ and $s_\times$ as order parameters for the free energy, and assume s_0 is a fixed number.

i. Determine the transformation of the 2×2 strain matrix, and thus s_+ and $s_\times$, under D_8.[2]

ii. Write down the most general free energy $F(s_+, s_\times)$ compatible with D_8 symmetry.

iii. Find the possible patterns of symmetry breaking, and find the invariant subgroups for each. Express each subgroup in terms of its generators, as they are defined in (8).

iv. Check that the number of cosets matches the number of inequivalent minima in each case above.

15 **Problem 5 (Grand unification):** Show the following groups are (as denoted) subgroups of each other:[3]

$$\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{U}(1) \le \mathrm{SU}(5) \le \mathrm{SO}(10). \tag{10}$$

This chain of subgroups is useful in theories of particle physics that try to unify all of the forces (strong, weak, electromagnetic) into a "single" force.

[2] *Hint:* How does a 2×2 matrix transform, if it acts on vectors that transform as (8)?

[3] *Hint:* It might be helpful to first think about why $\mathrm{U}(2) \times \mathrm{U}(3) \le \mathrm{U}(5)$, and then restrict to $\mathrm{SU}(5)$. Note that $\mathrm{U}(1)$ is the group of multiplication of complex numbers with $|z| = 1$.