# Homework 6

- **Due:** 11:59 PM, March 1. Submit electronically on Canvas.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

25 **Problem 1:** Consider 3 vectors (which transform in the **3**, or spin-1 irrep) of SO(3): $A_i$, $B_i$ and $C_i$.

i. Evaluate $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$, for SO(3).

ii. Using Einstein's tensor notation, write down all possible ways to get a scalar (**1** under SO(3)) by combining the 3 vectors above in an SO(3) invariant way.

iii. Using Einstein's tensor notation, write down all possible ways to get a vector (**3** under SO(3)) by combining the 3 vectors above in an SO(3) invariant way.

**Problem 2 (The hyperfine interaction):** A textbook problem in quantum mechanics involves the interaction of the electron and proton spins in the hydrogen atom. Let $\mathbf{S}_1$ denote the electron spin, and $\mathbf{S}_2$ the proton spin; each is spin-$\frac{1}{2}$. Consider the Hamiltonian

$$H = A\mathbf{S}_1\cdot\mathbf{S}_2 + B(S_{1z}+S_{2z}). \tag{1}$$

10 (a) Begin by setting $B=0$.

i. Write down the eigenvalues and eigenvectors of $H$. You do not need to derive them (you may even have memorized them!).

ii. Explain, in terms of group/representation theory, why all textbooks solve this problem the same way: first writing $H$ in terms of the total spin operator, $\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2$.

iii. Explain why the spectrum of $H$ is as generic as possible (given symmetry).

15 (b) Now, consider the case $B\neq 0$.

i. What are the eigenvalues and eigenvectors of $H$?

ii. What is the symmetry group under which $H$ is invariant?[1]

iii. Explain, in terms of group/representation theory, why there is no longer any degeneracy in $H$.

20 **Problem 3:** Consider the irreducible representations of O(2), which we labeled as $U_0^\pm$ and $R_1, R_2,\ldots$ in Lecture 12. Letting $R_0 := U_0^+\oplus U_0^-$, determine $R_n\otimes R_m$ for any $n,m\geq 0$.[2]

[1] *Hint:* $[H,S_z]=0$.

[2] *Hint:* Let $M_\theta\in\mathrm{O}(2)$ denote the rotation by $\theta$ degrees. Evaluate characters $\chi^{(R)}(M_\theta)$ for all of the irreps, and then for the product representation $R_n\otimes R_m$.

**Problem 4 ($D_{12}$-invariant polynomials):** Consider the vector space $\mathcal{P}$ of all polynomials in two variables $x$ and $y$:

$$\mathcal{P} = \mathrm{span}(1, x, y, x^2, xy, y^2, \ldots). \tag{2}$$

Consider the natural action of the dihedral group $D_{12}$ on this space, with reflection $s$ changing the sign of $y$, and rotation $r$ corresponding to a 60° coordinate rotation.

25 (a) Let $\mathcal{P}_n$ denote the subspace of $n^{\text{th}}$ order polynomials, of dimension $n+1$. Note that

$$\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots . \tag{3}$$

How do these vector spaces break up into irreps of $D_{12}$?

i. Explain why $\mathcal{P}_n$ is (a vector space corresponding to) a representation of O(2).

ii. Define $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$. Show that under a generic rotation around the origin by angle $\theta$, $z \to z\mathrm{e}^{\mathrm{i}\theta}$ and $\bar{z} \to \bar{z}\mathrm{e}^{-\mathrm{i}\theta}$.

iii. What does a reflection in O(2) [$y \to -y$, $x \to x$] do to $z$ and $\bar{z}$?

iv. Write the linearly independent vectors (polynomials) in $\mathcal{P}_n$ in terms of $z$ and $\bar{z}$.

v. Let $M_\theta$ denote a rotation by angle $\theta$, and $s$ denote the reflection operation in O(2). Explain why the following is a representation of O(2) isomorphic to $R_n$ (defined in Lecture 12):[3]

$$M_\theta = \begin{pmatrix} \mathrm{e}^{\mathrm{i}n\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}n\theta} \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4}$$

(You do not need to check that the multiplication rules of O(2) are obeyed; they are!)

vi. Explain how $\mathcal{P}_n$ decomposes into irreps of O(2).

vii. Explain how $\mathcal{P}_n$ decomposes into irreps of $D_{12}$.

15 (b) Now, consider a two dimensional isotropic quantum harmonic oscillator

$$H_0 = \frac{p_x^2 + p_y^2}{2} + \frac{1}{2}\left(x^2 + y^2\right). \tag{5}$$

Here we are working in units where $\hbar = m = \omega = 1$. Recall that the eigenfunctions of $H_0$ can be written (in part) in terms of Hermite polynomials.

i. Explain why $H$ is invariant under a "large" symmetry group $G$, with $D_{12} \le G$.

ii. Consider the 4 eigenvectors of $H$ with energy $E = 4$. Show that they may be chosen to form irreps of $D_{12}$, and give explicit expressions for the eigenvectors that transform in appropriate irreps (they do not need to be normalized).

iii. Suppose that we perturb $H_0$ with a generic small perturbation $H'$: $H = H_0 + \delta H'$, with $\delta \ll 1$. Assume that $H'$ is also $D_{12}$ invariant. Doing no further calculations, explain how the degeneracy of the $E = 4$ level will *generically* be lifted, and find (at leading order in $\delta$) the eigenvectors of $H$.

15 **Problem 5 (Crystal field splitting in a cubic crystal):** An atom has 5 degenerate d-states (with angular momentum $l = 2$). Now consider this atom when placed in a cubic crystal with symmetry group $O_h = S_4 \times \mathbb{Z}_2$, as discussed in Lecture 9. Based on symmetry considerations alone, into how many distinct energy levels do you expect these d-states to be split?[4]

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[3] *Hint:* Check characters!

[4] *Hint:* What is $\mathbf{3} \otimes \mathbf{3}$ in SO(3)? What about in $S_4$?