PHYS 5040: Algebra and Topology in Physics **Spring 2021**

Homework 7

- **Due:** 11:59 PM, March 8. Submit electronically on Canvas.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

20 **Problem 1:** Find 5 distinct irreducible representations of SO(6). What tensors do they correspond to? What is the dimension of each?

Problem 2 (Hidden symmetry of the quantum harmonic oscillator): Consider the three dimensional quantum harmonic oscillator

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2 + z^2\right). \tag{1}$$

10 (a) Let's review the solution of this problem. We begin by writing H in terms of raising and lowering operators:

$$H = \hbar\omega\left(\frac{3}{2} + a_i^\dagger a_i\right). \tag{2}$$

We are using the Einstein summation convention on repeated indices here ($i \in \{x, y, z\}$).

i. Find the energy levels E_n ($n = 0, 1, 2, \ldots$) of H.

ii. Show that the degeneracy of energy level E_n is

$$D(n) = \frac{(n+1)(n+2)}{2} \tag{3}$$

20 (b) The degeneracy $D(n)$ is much larger than what could be predicted by the rotational symmetry (SO(3)) of this problem on its own. Is there a hidden symmetry group responsible for this large degeneracy?

i. For which matrices U_{ij} does the transformation

$$a_i \to U_{ij} a_j, \tag{4a}$$

$$a_i^\dagger \to a_j^\dagger U_{ji}^\dagger \tag{4b}$$

leave H invariant? What is the resulting symmetry group of H?

ii. Use representation theory to account for the degeneracy $D(n)$ found above. You do not need to show in detail how the wave functions of the oscillator fit into the irreps, only that the degeneracy $D(n)$ is precisely accounted for by the usual counting of symmetry-enforced degeneracies.

Problem 3 (Mesons and baryons): The earliest elementary particles that were discovered can be thought of as consisting of combinations of 3 different flavors of quarks or antiquarks. These 3 different kinds of quarks: up (u), down (d) and strange (s), can be thought of as forming a $\mathbf{3} = (1,0)$ irrep of some approximate SU(3) flavor symmetry. The antiquarks are similar, but form a $\overline{\mathbf{3}} = (0,1)$ irrep. Although this is not an exact symmetry of nature, it proved quite valuable historically to treat this symmetry as approximate and to organize the elementary particles according to irreps of SU(3). In this problem, we will get some idea of how this all worked out.

In what follows, it will be useful to use the fact that both quarks and antiquarks are spin-$\frac{1}{2}$ fermions ($\mathbf{2}$ of SU(2) rotation symmetry). It will also be useful to keep track of the electric charges of the quarks:

$$q_\mathrm{u} = \frac{2}{3}e, \quad q_\mathrm{d} = q_\mathrm{s} = -\frac{1}{3}e. \tag{5}$$

Here e is the charge of the proton. Antiquarks of a given flavor have opposite electric charge to the corresponding quark flavor.

15 (a) The mesons correspond to a combination of a quark and antiquark.

i. Evaluate $\mathbf{3} \otimes \overline{\mathbf{3}}$ to determine the possible flavor SU(3) irreps of the meson particles.

ii. For each irrep, determine how many particles there can be of any given electric charge.[1]

iii. In addition to charge and SU(3) irrep, each meson also has a physical spin. Multiply together the spin irreps of SU(2) to determine the allowed values of spin for a meson.

25 (b) Baryons are particles consisting of three quarks. The baryonic wave function is rather complicated, consisting of four components:

$$|\text{baryon}\rangle = |\text{position}\rangle \otimes |\text{color}\rangle \otimes |\text{flavor}\rangle \otimes |\text{spin}\rangle \tag{6}$$

The 3 quarks making up a baryonic particle are indistinguishable fermions (therefore, the overall wave function should be antisymmetric under exchange of any two quarks). The position-space wave function $|\text{position}\rangle$ will be symmetric in a particle's ground state, while $|\text{color}\rangle$ will be antisymmetric. Therefore $|\text{flavor}\rangle \otimes |\text{spin}\rangle$ must be a symmetric wave function under particle exchange.

i. Show that baryonic particles must have either total spin $j = \frac{3}{2}$ or $j = \frac{1}{2}$.

ii. Suppose that we have a baryon of spin-$\frac{3}{2}$. By constructing the state with maximal z-spin explicitly (in the "uncoupled" basis of 3 quark spins), conclude that $|\text{spin}\rangle$ should be a symmetric wave function under the quark exchange operation, when $j = \frac{3}{2}$.

iii. Show that the possible SU(3) irreps of a baryon are $\mathbf{10} = (3,0)$, $\mathbf{8} = (1,1)$, or $\mathbf{1} = (0,0)$.

iv. Conclude that there will be a set of 10 baryonic particles of spin-$\frac{3}{2}$. Determine the possible charges of these particles, and how many particles must have each charge.

v. Show that the $\mathbf{1}$'s flavor wave functions must be fully antisymmetric. You can do this with a one sentence argument!

vi. Show that it is not possible to have a baryonic particle in the $\mathbf{1}$ irrep of SU(3). Again, there is a very short argument!

vii. There is also a family of spin-$\frac{1}{2}$ baryons in the $\mathbf{8}$ of SU(3). This is a bit more tedious to show, so you do not need to do it. What are the possible electric charges of these particles?[2]

[1] *Hint:* The combination $\mathrm{u}\bar{\mathrm{u}}$ has charge $q_\mathrm{u} - q_\mathrm{u}$, $\mathrm{u}\bar{\mathrm{d}}$ has charge $q_\mathrm{u} - q_\mathrm{d}$, etc.

[2] *Hint:* If you don't want to construct a basis for an $\mathbf{8}$ inside $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$, figure out the charges of all 27 "particles" possible within $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$, and subtract out the charges of particles that transform in the $\mathbf{10}$ and $\mathbf{1}$ irreps.

15 **Problem 4 (Self-dual and anti-self-dual tensors):** Consider the group $\mathrm{SO}(2n)$ with n a positive integer. Consider the representation $\mathcal{A}$ consisting of rank-n fully antisymmetric tensors

$$A_{i_1 i_2 i_3 \cdots i_n} = -A_{i_2 i_1 i_3 \cdots i_n} = \cdots = -A_{i_n i_2 i_3 \cdots i_1}. \tag{7}$$

i. Define

$$B_{i_1 \cdots i_n} = \frac{1}{n!} \epsilon_{i_1 \cdots i_n j_1 \cdots j_n} A_{j_1 \cdots j_n}. \tag{8}$$

Show that

$$A_{i_1 \cdots i_n} = \frac{(-1)^n}{n!} \epsilon_{i_1 \cdots i_n j_1 \cdots j_n} B_{j_1 \cdots j_n}. \tag{9}$$

ii. Conclude that for an appropriate choice of $\alpha = 1, \mathrm{i}$ which depends on n, that

$$\frac{1}{n!} \epsilon_{i_1 \cdots i_n j_1 \cdots j_n} \left(A_{j_1 \cdots j_n} \pm \alpha B_{j_1 \cdots j_n} \right) = \pm \alpha^{-1} \left(A_{i_1 \cdots i_n} \pm \alpha B_{i_1 \cdots i_n} \right) \tag{10}$$

iii. Conclude that $\mathcal{A}$ is a reducible representation, that splits up into two representations corresponding to self-dual tensors $(+)$ and anti-self-dual tensors $(-)$. The self-dual and anti-self-dual representations turn out to be irreducible.

iv. What is the dimension of the self-dual representation?

15 **Problem 5 (Chiral anomaly):** In four dimensional quantum field theory, it is possible to have a theory of chiral fermions, coupled to gauge fields (generalizations of electromagnetic fields). These theories turn out to be "sick" (classical symmetries fail to be symmetries of the full quantum theory) if a certain anomaly coefficient, associated with the symmetry group G of the system, does not vanish.

The group theoretic essence of the chiral anomaly boils down to the following simple calculation. Let T^a denote the generators of the group G, which we'll assume is a Lie group, in whichever (possibly reducible) representation R is associated with the matter content. Then, the chiral anomaly coefficients are defined as

$$\mathcal{A}^{abc} := \mathrm{tr}(T^a T^b T^c + T^a T^c T^b). \tag{11}$$

We want to have all $\mathcal{A}^{abc} = 0$ to have an anomaly free theory.

Consider a gauge theory with group $G = \mathrm{SO}(n)$. Find all possible n for which the chiral anomaly may be non-vanishing.[3]

The fact that the chiral anomalies necessarily vanish for so many of the Lie groups $\mathrm{SO}(n)$ makes them very popular in studies of grand unification in particle physics.

[3] *Hint:* To do this, I suggest you write the generators in terms of $T^{ij} = -T^{ji}$. Then $\mathcal{A}$ is a 6 index object, with certain symmetries required upon exchanging the 6 indices. Since only δ^{ij} and the Levi-Civita tensor are $\mathrm{SO}(n)$ invariant, it must be possible to express $\mathcal{A}$ in terms of these tensors alone. When is that possible?