PHYS 5040: Algebra and Topology in Physics Spring 2021

Homework 8

- **Due:** 11:59 PM, March 15. Submit electronically on Canvas.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

25 **Problem 1 (Nematic liquid crystals):** A nematic liquid crystal molecule can be thought of as a rod whose "head" and "tail" are equivalent. Upon fixing the center of the rod, the space of all possible configurations of the nematic corresponds to the set of all lines passing through the origin in three dimensional space, called $\mathbb{R}\mathrm{P}^2$.

i. Explain why
$$\mathbb{R}\mathrm{P}^2 := (\mathbb{R}^3 - \{\mathbf{0}\})/\sim, \tag{1}$$
where $\mathbf{0}$ denotes the origin of $\mathbb{R}^3$, and the equivalence relation $\sim$ identifies $\mathbf{x} \sim \lambda\mathbf{x}$ for any $\lambda \neq 0$.

ii. Explain why $\mathbb{R}\mathrm{P}^2$ can *also* be thought of as the sphere S^2, with opposite points identified.[1]

iii. Show how to build $\mathbb{R}\mathrm{P}^2$ as either a CW complex or a simplicial complex, whichever you prefer.[2]

Problem 2 (From polyhedra to solid-state physics and chemistry): A regular polyhedron is a two-dimensional shape, made out of M identical regular polygons of n sides. Here M and n are, a priori, arbitrary positive integers. Let us also demand that in our regular polyhedron, any vertex can (by appropriate rotations) be mapped to any other vertex.

25 (a) Platonic solids are two dimensional surfaces with Euler characteristic $\chi = 2$, obeying all of the above properties. Prove that there are 5 Platonic solids.[3] What are they? (If you can't figure this out from first principles, at least look up the answer for later parts of the problem!)

15 (b) Now, let's turn to solid-state physics. How many regular crystalline lattices (periodic tilings of the plane) exist in two dimensions, which can be built out of regular polygons? Find the answer by generalizing the method of part (a) to the 2-torus.

45 (c) The point group of a non-linear molecule in three spatial dimensions (or of the unit cells – periodic motifs that make up a solid) must correspond to a finite subgroup $G \leq \mathrm{SO}(3)$. G corresponds to the rotations that leave the molecule invariant. Remarkably, we can completely classify all possible subgroups G!

Only basic group theory from Lectures 1-4 are needed for this part, together with the fact (which you do not need to prove) that all non-identity elements in $\mathrm{SO}(3)$ can be thought of as rotations by angle $0 < \theta < 2\pi$ around an axis (line passing through the origin of three-dimensional space). Let S^2 be the unit sphere ($x^2 + y^2 + z^2 = 1$); $\mathrm{SO}(3)$ acts on S^2 by rotating the coordinates in the standard way!

Let n denote the finite number of elements in the subgroup G.

[1] *Hint:* $x^2 + y^2 + z^2 = 1$ is an embedding of S^2 into $\mathbb{R}^3$. What does $\sim$ do to this embedding?

[2] *Hint:* First, think about how to build a CW complex for S^2 such that two opposite points on the sphere always correspond to different cells.

[3] *Hint:* First show that $V = \alpha F$ and $E = \beta F$; how can you constrain α and β?

i. Let $g \in G$ be a non-identity element. Show that there are exactly two points on the sphere which are left invariant by the rotation g: in other words, $g \cdot p = p$ for $p \in \mathrm{S}^2$ has two solutions.

ii. Let V be the set of all points on S^2 that are left invariant by at least one non-trivial rotation in G. Show that the action of rotations in G is a well-defined group action on V: in other words, if $p \in V$ and $g \in G$, then $g \cdot p \in V$.[4]

iii. Consider a point p which is left fixed by some non-identity g; let H_p be the stabilizer group (or little group, to physicists) of point p. Show that H_p is isomorphic to $\mathbb{Z}_m$, for some integer m dividing n. We then say that p has order m.

iv. Let v_m be the number of points in V whose stabilizer group is isomorphic to $\mathbb{Z}_m$. Show that n/m must be an integer, and that n/m also divides v_m.[5]

v. By counting the number of non-trivial rotations in G by first summing together the number of rotations that leave points p of order m invariant, show that there exist non-negative integers $k_2, k_3, k_4, \ldots$ such that

$$2 - \frac{2}{n} = \sum_{m=2}^{\infty} \left(1 - \frac{1}{m}\right) k_m. \tag{2}$$

vi. Show that

$$1 < \sum_{m=2}^{\infty} k_m < 4. \tag{3}$$

vii. Noting that $k_m = 0$ if m does not divide n, conclude that there is only one infinite family of solutions to (2) with $\sum k = 2$: $k_q = 2\delta_{n,q}$, for any n. Explain why the resulting symmetry group G is isomorphic to $\mathbb{Z}_m$ (for $m \geq 2$).

viii. Construct an infinite family of solutions to (2) with $\sum k = 3$, where the resulting symmetry group G is isomorphic to D_{2m} ($m \geq 2$).

ix. Show that there are three *exceptional* solutions to (2) obeying $\sum k = 3$, which do not fall into the categories above: one with $n = 12$, one with $n = 24$, and one with $n = 60$. As part of your solution, rule out the existence of all other point groups.

This remarkable construction gives us a *complete* classification of all possible rotation symmetries in three dimensions! This is why books on chemistry and solids can exhaustively list all possible characters and irreducible representations for molecular orbitals, etc.

10 (d) The results of parts (a) and (c) are deeply interwined. Indeed, every Platonic solid has a point group (rotational symmetry group) G, not isomorphic to either $\mathbb{Z}_n$ or D_{2n} for any n. (Think about why this has to be the case!)

i. Prove that the point groups of the octahedron G_{oct} and the cube G_{cube} are isomorphic.[6]

A similar proof (which you do not need to show) reveals that the dodecahedron and icosahedron have the same point group, and that the tetrahedron can have its own point group.

ii. Argue that the 3 "exceptional" point groups from part (c) must be the point groups of the Platonic solids.

[4] *Hint:* Let $h \cdot p = p$. What group element can you *construct* that leaves $h \cdot p$ invariant?

[5] *Hint:* Let p be one of these points. How many different points in V can we find by evaluating $g \cdot p$? What do we know about each of these points? Lecture 4 may be useful.

[6] *Hint:* Prove that $G_{\text{oct}} \leq G_{\text{cube}}$ by showing that every rotation of an octahedron that leaves it invariant, must also leave a cube invariant. Where might this cube be, geometrically? Then, do the same thing with the role of the cube and octahedron exchanged.