PHYS 5040: Algebra and Topology in Physics **Spring 2021**

Homework 9

- **Due:** 11:59 PM, March 29. Submit electronically on Canvas.
- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

20 **Problem 1 (Thermodynamics):** The "infinitesimal" form of the first law of thermodynamics is sometimes written as

$$\mathrm{d}E = \delta Q - \delta W. \tag{1}$$

Here $\mathrm{d}E$ is the infinitesimal change in the internal energy E under a change in entropy $S \to S + \mathrm{d}S$ and volume $V \to V + \mathrm{d}V$. The infinitesimal change in the heat absorbed by the system is given by

$$\delta Q = T\mathrm{d}S, \tag{2}$$

while the mechanical work done by the system is

$$\delta W = P\mathrm{d}V. \tag{3}$$

Here T and P are temperature and pressure, respectively.

Using the language of differential forms, we can interpret $\mathrm{d}E$, δW and δQ as 1-forms on a manifold characterized by a *choice* of local coordinates, S and V.

1.1. In thermodynamics, the energy $E(S, V)$ is a well-defined (single-valued) function of S and V. Hence, we expect $\mathrm{d}E$ is an exact differential. Conclude formulas relating T and P to E, S and V.

1.2. Use the fact that $\mathrm{d}^2E = 0$ to obtain a thermodynamic Maxwell relation, relating T and P to each other.

1.3. Let γ be a *closed* thermodynamic cycle (a trajectory through the space of state variables, such as S and V). The total work done by the system is given by

$$W = \int_\gamma \delta W = \int_\gamma P\mathrm{d}V. \tag{4}$$

Generally, $W \neq 0$. In thermodynamics textbooks, it is often stated that δW and δQ are "inexact differentials". Mathematically, what does that imply?

1.4. In the language of differential forms, (1) does not depend on a choice of "coordinates" on the thermodynamic manifold. Explain why if we define the free energy

$$F = E - TS, \tag{5}$$

the exact differential $\mathrm{d}F$ leads us to a new "thermodynamics", where T and V become a natural choice of local coordinates. What constraints do we obtain on P and S?

15 **Problem 2:** Consider the manifold $M = \mathbb{R} \times \mathrm{S}^1$.

2.1. What well known shape is this manifold homeomorphic (topologically equivalent) to?

2.2. As we'll learn to show later, M is topologically non-trivial. Despite this fact, find a *single* coordinate chart that covers all of M.

20 **Problem 3:** Let ω_1 be a q-form and ω_2 be an r-form. Show that

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^q \omega_1 \wedge \mathrm{d}\omega_2. \tag{6}$$

Problem 4 (Higher form symmetries): A recent topic of some interest in theoretical physics is the existence of higher form symmetries.

15 **4A:** Let's begin by describing an "ordinary" (continuous) symmetry in the language of differential forms. Consider a theory with a single conserved charge, where the continuity equation is

$$\partial_t \rho + \partial_i J_i = 0. \tag{7}$$

4A.1. Define the current 1-form $J := \rho \mathrm{d}t + J_i \mathrm{d}x_i$. Write (7) using differential forms.

4A.2. Consider the dynamics of this conserved charge on spacetime manifold $M \times \mathbb{R}$, where M represents space (x_i) and $\mathbb{R}$ represents time t. Assuming that charge does not flow in/out at spatial infinity, use Stokes' Theorem to show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_M * J = 0. \tag{8}$$

What physical quantity does this integral represent?

4A.3. In quantum field theory, the main object of theoretical interest is the following:

$$Z[A] := \left\langle \exp\left[\int A \wedge * J\right] \right\rangle. \tag{9}$$

Here the 1-form A represents a classical background gauge field, while J is a quantum operator. $\langle \cdots \rangle$ represents a quantum expectation value (which plays no role in this problem). The integral is performed over the entire spacetime manifold. Argue that if charge is conserved (namely if (7) holds – do not fuss over J being an operator), then $Z[A]$ must be "gauge invariant": if λ is an arbitrary function (0-form),

$$Z[A] = Z[A + \mathrm{d}\lambda]. \tag{10}$$

15 **4B:** To construct a theory with a higher form symmetry, we can try to reverse the logic above. Suppose that you have a quantum theory which you can couple to a $(p+1)$-form A, such that (10) holds. We then say that we have a p-form symmetry.

4B.1. What object must λ be in order for this construction to make sense?

4B.2. If (9) holds, what must the object J be?

4B.3. What is the generalization of the conservation law (7)?

4B.4. Argue that there is an interesting natural generalization of (8), if we replace the integral over all of space M with an integral over some (possibly) lower dimensional closed surface Σ in M (namely, a surface Σ with boundary $\partial\Sigma$ the empty set). What dimension is Σ?

10 **4C:** As an interesting example of a higher form symmetry, we turn to ordinary electromagnetism. For simplicity, you can focus on electromagnetism in $3+1$ spacetime dimensions.

4C.1. If F represents the Maxwell tensor, show that $*F$ is the current of a 1-form symmetry.

4C.2. Physically interpret the conserved quantities associated with this 1-form symmmetry.[1]

Problem 5 (K3 surfaces): In string theory, a family of 4-dimensional manifolds called K3 surfaces have wide-ranging importance. In this problem, we will sketch out one construction of a K3 surface.

10 **5A:** We begin with a discussion of **complex projective space** $\mathbb{CP}^n$, which is defined to be

$$\mathbb{CP}^n := \left(\mathbb{C}^{n+1} - \mathbf{0}\right) / \sim \tag{11}$$

where

$$(z_1, \ldots, z_{n+1}) \sim \lambda(z_1, \ldots, z_{n+1}), \quad (\lambda \neq 0). \tag{12}$$

5A.1. Argue that $\mathbb{CP}^n$ is a manifold which admits an atlas with $n+1$ charts. What are they?

5A.2. What is the dimension of $\mathbb{CP}^n$?

5A.3. Explain why $\mathbb{CP}^1 = \mathrm{S}^2$.

15 **5B:** Now, consider the space $\mathbb{CP}^3$, which we have defined using the coordinates above.

5B.1. In one sentence, explain why the following equation is well-defined in $\mathbb{CP}^3$:

$$0 = z_1^n + z_2^n + z_3^n + z_4^n. \tag{13}$$

Let Σ denote the subset of $\mathbb{CP}^3$ obeying this equation.

5B.2. Suppose that $z_4 \neq 0$. Define $w_i = z_i/z_4$ $(i = 1, 2, 3)$. Show that

$$\sum_{i=1}^{3} w_i^{n-1} \mathrm{d}w_i = 0. \tag{14}$$

5B.3. Consider the following differential form:

$$\omega = \frac{\mathrm{d}w_1 \wedge \mathrm{d}w_2}{w_3^{n-1}} = \frac{\mathrm{d}w_2 \wedge \mathrm{d}w_3}{w_1^{n-1}} = \frac{\mathrm{d}w_3 \wedge \mathrm{d}w_1}{w_2^{n-1}}. \tag{15}$$

Explain why all 3 expressions for ω are equivalent (if $w_{1,2,3} \neq 0$). Conclude that even at points where one (or two) of the $w_i = 0$, the 2-form ω is still well-defined and is not singular.

5B.4. Now, consider the subset of Σ where both z_1 and z_4 are non-vanishing. We can use either the coordinate chart above, *or* we can define $v_i = z_i/z_1$ $(i = 2, 3, 4)$. Show how to convert between w_i and v_i coordinates.

5B.5. Suppose that we try to extend ω from the patch where $z_4 \neq 0$ to the patch where $z_1 \neq 0$. When doing so, we need to make sure that as $v_4 \to \infty$ in the new patch, we can switch ω to its definition in the old patch, without any singularities. Show that this is only possible if $n = 4$.

Continuing this construction gives us a globally defined 2-form ω (defined entirely in terms of holomorphic coordinates z_i, not $\bar{z}_i$) which is nowhere zero/singular. This is a sufficient criterion for a K3 surface.

What is impressive about these constructions is that they give us some intuition for how to build up complicated manifolds, with non-trivial differential forms, using only "basic" operations.

[1] *Hint:* The answer is either electric fluxes and/or magnetic fluxes – through what shapes?