PHYS 5040
Algebra and Topology in Physics
Spring 2021

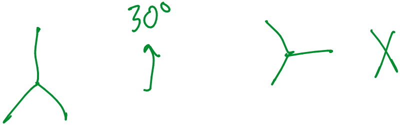
Lecture 1

January 14

1 Draw the molecule BH_3. What are the "symmetries" of this molecule?

reflection:

rotation by 120°:



- symmetry = [set of] transforms that leave object invariant

- you can "multiply" transformations together

2 Symmetries form a group. What properties should a group have?

group G is a set $\{g_1, g_2, \dots\}$ w/ "multiplication" (binary operation)

1) closure: $g_1 \times g_2 \in G$

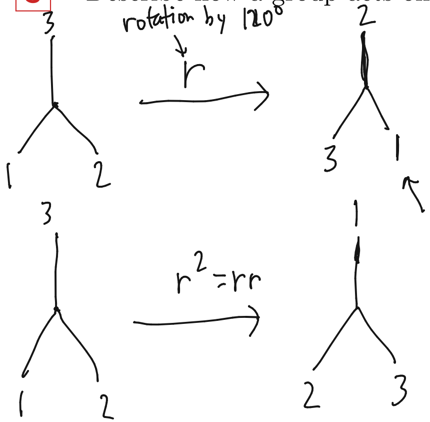
2) associative: $(g_1 \times g_2) \times g_3 = g_1 \times (g_2 \times g_3) = g_1 g_2 g_3$

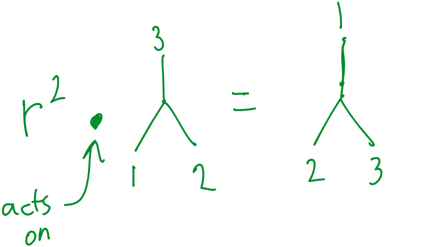
3) identity: $\underset{\text{there exists}}{\exists}\ \underset{\text{identity}}{1} \overset{\text{element of}}{\in} G$ obeying $1g = g1 = g$, $\underset{\text{for all}}{\forall}\, g \in G$

4) inverses: $\forall g \in G$, $\exists\ \overset{\text{inverse}}{g^{-1}}$ obeys $g^{-1}g = gg^{-1} = 1$

Proposition:
- 1 is unique
- g^{-1} is unique

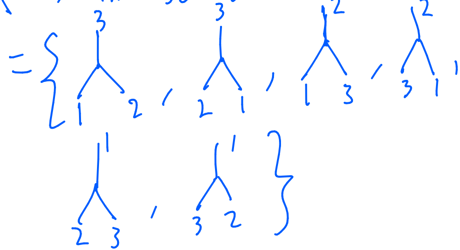
3 Describe how a group acts on a set.





G is the symmetry group of BH_3

X is the set of molecule configs



$$(g_1 g_2)\cdot \underset{\in X}{x} = g_1 \cdot (g_2 \cdot x)$$

4 What is the symmetry group of BH_3?

$1 \cdot$ (c; a, b) $=$ (c; a, b)

non-Abelian group $=$ $g_1 g_2 \neq g_2 g_1$ holds $\exists\, g_1, g_2 \in G$

Abelian group: $g_1 g_2 = g_2 g_1$

reflection:

$s \cdot$ (c; a, b) $=$ (c; b, a)

$[s^2 = 1]$ $s^2 \cdot$ (c; a, b) $=$ (c; a, b)

rotation by $120°$:

$r \cdot$ (c; a, b) $=$ (b; c, a)

$r^2 \cdot$ (c; a, b) $=$ (a; b, c)

$r^3 \cdot$ (c; a, b) $=$ (c; a, b)

$(rs) \cdot$ (c; a, b) $= r \cdot$ (c; b, a) $=$ (a; c, b)

$sr \neq rs$

$sr \cdot$ (c; a, b) $=$ (b; a, c)

$[r^3 = 1]$ $s \cdot (r \cdot$ (c; a, b)$) = s \cdot ($(b; c, a)$)$

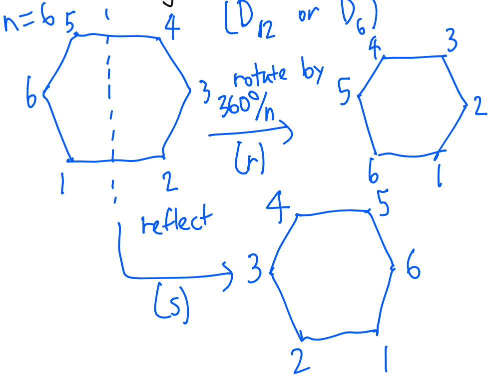
5 Generalize, and describe the dihedral groups D_{2n}.

dihedral groups = symmetry group of regular polygon

D_{2n} = n-gon (n sides)

[Zee: D_n]

n=6 [D_{12} or D_6]



identity

$$D_{2n} = \{1, r, r^2, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}$$

$r^n = 1$, $s^2 = 1$

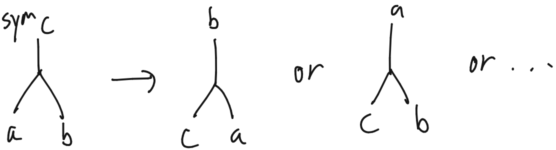
$rs = sr^{-1} = sr^{n-1}$

E.g. (n=6): $r(sr^4) = (rs)r^4 = sr^5 r^4 = sr^3$

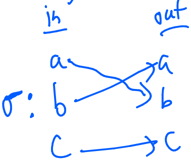
symmetry group of BH_3 is D_6 (or D_3 if Zee)

6 Describe the symmetries of BH_3 as a permutation group.

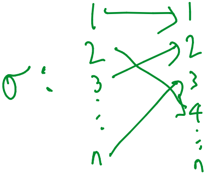
BH_3's sym
$\downarrow$
g ·



BH_3 symmetry group = group of permutations of 3 objects, S_3.



group of permutations acting on n objects $= S_n$

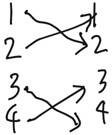


group "multiplication" = function composition

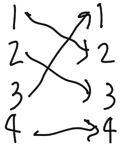
7 How do we multiply together permutations?

Example: $(n=4)$

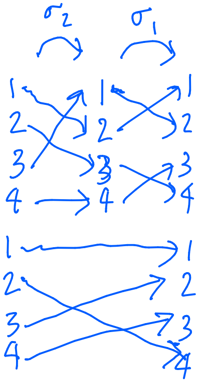
σ_1:



σ_2:



$\sigma_1\sigma_2$:



Cycle notation:

σ_1: $1\ 2\quad 3\ 4 = (1\ 2)(3\ 4)$

σ_2: $1\ 2\ 3\quad 4 = (1\ 2\ 3)$

$\sigma_1\sigma_2 = (2\ 4\ 3) = (4\ 3\ 2)$

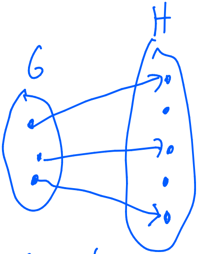
8 Define a group homomorphism.

homomorphism: map btwn groups preserves multiplication:

$$\underbrace{\varphi : G \to H}_{G,\ H \text{ groups}}$$

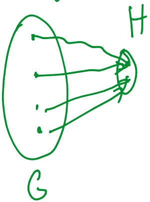
$$\varphi(\underbrace{g_1 g_2}_{\text{mult in } G}) = \underbrace{\varphi(g_1)\varphi(g_2)}_{\text{mult in } H}$$

injective:

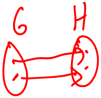


$\varphi(g_1) = \varphi(g_2) \xrightarrow{\text{implies}} g_1 = g_2$

surjective:



$\forall h \in H$,
$\exists g$ s.t. $\varphi(g) = h$



bijective:
injective +
surjective
(one-to-one)

invertible

9 Define a group isomorphism. Explain why $D_6 = S_3$.

bijective (invertible) homomorphism = isomorphism

$$\begin{tikzcd} S_3 \arrow[r, bend left, "\psi^{-1}"] & \text{symmetry group of } BH_3 \arrow[l, bend left, "\psi"] \arrow[r, bend left, "\phi"] & D_6 \arrow[l, bend left, "\phi^{-1}"] \end{tikzcd}$$

function composition

$$\phi \circ \psi^{-1} : S_3 \xrightarrow{BH_3 \text{ group}} D_6$$ is an isomorphism

$$D_6 \longrightarrow S_3$$ use $\psi \circ \phi^{-1}$

when two groups are isomorphic, for all intents and purposes, they are "equal" $D_6 = S_3$