PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 10

February 16

1 What is a Lie group?

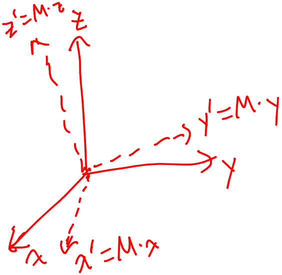
Lie group = group that is also $\underbrace{\text{"continuous"}}_{\text{a manifold}}$

Example 1: translations in 1d: $(x_1, x_2) \longrightarrow (x_1+a, x_2+a)$, $a\in\mathbb{R}$

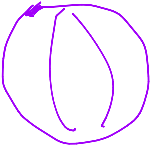
non-compact

3x3 real matrices

Example 2: 3d rotations: $SO(3)=\{M\in\mathbb{R}^{3\times 3} : \det(M)=1, M^T M=1\}$



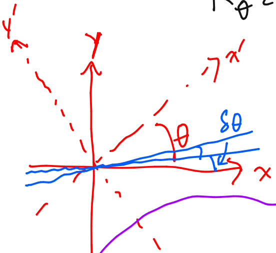
SO(3): compact



2 What is the generator of SO(2)?

Example: $SO(2) = \{ M \in \mathbb{R}^{2\times 2} : \det(M) = 1,\ M^T M = 1 \}$

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad 0 \le \theta < 2\pi$$



$$\left(R_{\delta\theta}\right)^{\frac{\theta}{\delta\theta}} = R_\theta$$

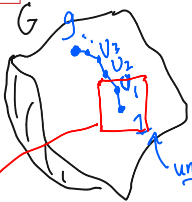
$$R_{\delta\theta} \approx \begin{pmatrix} 1 - \frac{\delta\theta^2}{2} + \cdots & -\delta\theta \\ \delta\theta & 1 - \frac{\delta\theta^2}{2} \end{pmatrix}$$

$$\approx 1 + \delta\theta \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\substack{T \\ \text{(generator)}}}$$

$$\left[1 + \delta\theta \cdot T\right]^{\frac{\theta}{\delta\theta}} \approx \left(\exp\left[\delta\theta \cdot T\right]\right)^{\frac{\theta}{\delta\theta}} = \exp(\theta \cdot T)$$

$$\exp(A) = 1 + A + \cancel{\tfrac{1}{2} A^2} + \cancel{\tfrac{1}{3!} A^3} + \ldots$$

3 Argue that all Lie groups can be built from generators.



unique element

$$g = (1+\delta U_N)\cdots(1+\delta U_2)(1+\delta U_1)\mathbb{1}$$

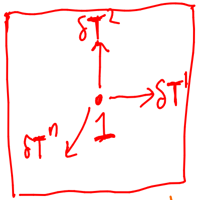
$$\left[N \sim \frac{1}{\delta}\right]$$

$$g = \exp(\delta U_N)\exp(\delta U_{N-1})\cdots\exp(\delta U_1)\mathbb{1}$$

which δU's are allowed?

n-dimensional: (locally $\mathbb{R}^n$)

zoom in



near the identity...

generators

$$\exp(\delta U_n) = 1 + \underbrace{\delta\theta_n^a}_{\text{infinitesimal}} T^a$$

← Einstein sum

Choose T^a to be lin. ind.

$\mathrm{tr}(T^a T^b) = \delta^{ab}$

$$\exp(\delta\theta_1^a T^a)\exp(\delta\theta_2^a T^a)$$
$$= (1+\delta\theta_1^a T^a)(1+\delta\theta_2^a T^a)$$
$$= 1 + (\delta\theta_1^a + \delta\theta_2^a)T^a \approx \exp((\delta\theta_1^a + \delta\theta_2^a)T^a)$$

4 What is a Lie algebra? Define the structure constants.

$$\exp(-\delta\alpha^a T^a)\exp(\delta\theta^a T^a)\exp(\delta\alpha^a T^a) \stackrel{?}{=} ?$$

let's expand to $\mathcal{O}(\delta\alpha^a)$, keep $\mathcal{O}(\delta\theta\,\delta\alpha)$

$$(1-\delta\alpha^a T^a)(1+\delta\theta^a T^a)(1+\delta\alpha^a T^a)$$

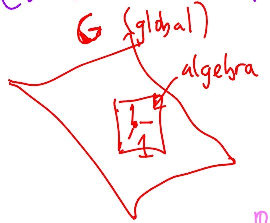
$$= 1 + (\cancel{-\delta\alpha^a} + \delta\theta^a + \cancel{\delta\alpha^a})T^a + \underbrace{\delta\theta^a\delta\alpha^b(T^aT^b - T^bT^a)}_{\text{must be of the form } \delta\phi^a T^a}$$

$$= 1 + \delta\tilde{\theta}^a T^a$$ [b/c both group elements close to identity]

Conclusion: $T^aT^b - T^bT^a = \boxed{[T^a, T^b] = f^{abc}T^c}$

Lie algebra of G, $\mathfrak{g}$

f^{abc} = structure constants



e.g. Lie algebra for $SO(2)$ has one element T...

for $\mathbb{R}$ also has one element [both Id]

5 Describe the Lie group $\mathrm{SO}(n)$ and its corresponding algebra $\mathfrak{so}(n)$.

special orthogonal: $SO(n) = \{ M \in \mathbb{R}^{n \times n} : \det(M) = 1, \ M^T M = 1 \}$

Step 1: What are generators? $M = \mathbb{1} + \delta\theta^a T^a$

$1 = \det(\mathbb{1} + \delta\theta^a T^a) = 1 + \mathrm{tr}(\delta\theta^a T^a)$: since holds $\forall\, \delta\theta^a \ldots$ $\mathrm{tr}(T^a) = 0$ → redundant!

$$\mathbb{1} = (\mathbb{1} + \delta\theta^a (T^a)^T)(\mathbb{1} + \delta\theta^a T^a) = \mathbb{1} + \delta\theta^a \underbrace{((T^a)^T + T^a)} + \mathcal{O}(\delta\theta^2)$$

$\Rightarrow T^a$ must be antisymmetric

natural basis for antisym:

$a = ij$

$$T_{ij} = -T_{ji} = e_i e_j^T - e_j e_i^T$$

if $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix}$ 1 in ith position

$$[T_{ij}, T_{k\ell}] = (e_i e_j^T - e_j e_i^T)(e_k e_\ell^T - e_\ell e_k^T) - (k\ell)(ij)$$

$$= e_i \underbrace{e_j^T e_k}_{e_j \cdot e_k = \delta_{jk}} e_\ell^T - e_j e_i^T e_k e_\ell^T + \ldots$$

$$= e_i e_\ell^T \delta_{jk} - e_j e_\ell^T \delta_{ik} + \ldots$$

$$= \delta_{jk} T_{i\ell} - \delta_{ik} T_{j\ell} - \delta_{j\ell} T_{ik} + \delta_{i\ell} T_{jk}$$

$\boxed{6}$ Describe the classification of all simple Lie algebras.

simple Lie algebra: can't decompose algebra as $T^a = \{A^i, B^\alpha\}$

NOT simple $\Bigg\{$
$$[A^i, A^j] = f_A^{ijk} A^k \qquad [B^\alpha, B^\beta] = f_B^{\alpha\beta\gamma} B^\gamma$$
$$[A^i, B^\alpha] = 0$$

special unitary: $\boxed{su(n)}$ $n \geq 2$

orthogonal: $\boxed{so(n)}$ unique when $n \geq 7$ (Zee VII)

unitary-symplectic: $sp(2n)$ $n \geq 2$

exceptional: e_6, e_7, $\boxed{e_8}$, f_4, g_2