PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 11

February 18

1 What is the Lie group SO(3)? What is its algebra?

$$SO(3) = \{ M \in \mathbb{R}^{3\times 3} \; ; \; \det(M) = 1, \; M^T M = \mathbb{1} \}$$

3x3 matrix w/ real coeff

rotation group in 3 dimension

classical example: Lagrangian for (non-rel.) universe

$$L = \sum_i \frac{1}{2} m \dot{\vec{x}}_i^{\,2} - \sum_{i<j} V_{ij}(|\vec{x}_i - \vec{x}_j|) \; :$$

L invariant if $\vec{x}_i \longmapsto M\vec{x}_i$

since length preserved

$$(M\vec{a} \cdot M\vec{a}) = \vec{a}^T \cancel{M^T M} \vec{a} = \vec{a} \cdot \vec{a}$$

Lie algebra so(3): generators

$$M = \mathbb{1} + \delta\theta^a T^a$$

last time: T^a's are antisymmetric

$$T^{xy} = T^z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad T^{zx} = T^y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad T^{yz} = T^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$[T^x, T^y] = T^z; \quad [T^y, T^z] = T^x; \quad [T^z, T^x] = T^y$$

angular momentum

2 What is the Lie group SU(2)? What is its algebra?

$$SU(2) = \{ M \in \mathbb{C}^{2\times 2} : \det(M) = 1, \quad M^\dagger M = 1 \}$$

What's the Lie algebra? $M = 1 + \delta\theta^a T^a$ ← unknown generators

$$\det(M) = \det(1 + \delta\theta^a T^a) = 1 + \delta\theta^a \operatorname{tr}(T^a) = 1 : \quad \operatorname{tr}(T^a) = 0$$

$$(1 + \overline{\delta\theta^a} (T^a)^\dagger)(1 + \delta\theta^b T^b) \approx 1 + \underbrace{T^a \delta\theta^a + (T^a)^\dagger \overline{\delta\theta^a}}_{=0} = 1$$

$=0$: if $\delta\theta^a \neq 0 \ldots$

(no sum on a) $\rightarrow T^a \delta\theta^a = -\overline{\delta\theta^a} (T^a)^\dagger$

$(T^a)^\dagger = T^a$ and $\delta\theta^a = i\,\delta\phi^a$ ← real

3 traceless Hermitian 2×2's ... Pauli:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Lie algebra:

$$[\sigma^x, \sigma^y] = 2i\sigma^z$$
$$[\sigma^y, \sigma^z] = 2i\sigma^x$$
$$[\sigma^z, \sigma^x] = 2i\sigma^y$$

up to factor of $2i$...

$$T^x = \frac{1}{2i}\sigma^x, \text{ etc...}$$

$$[T^x, T^y] = T^z \ldots$$

exactly same as $so(3)$

3 Review the "simultaneous diagonalization" of the angular momentum algebra.

$[J_x, J_y] = i\cancel{\hbar}^{1} J_z$ $\quad [J_z, J_x] = i\hbar J_y$ $\quad [J_y, J_z] = i\hbar J_x$

angular momentum

be diagonal

QM "solution" of this problem:

$J^2 = J_x^2 + J_y^2 + J_z^2 \ldots$ $\quad [J^2, J_x] = 0 = [J^2, J_y] = \underbrace{[J^2, J_z]}_{\text{diag. simultaneously}}$

$J_\pm = J_x \pm i J_y$

$[J_z, J_\pm] = i J_\pm$

if $J_z|\psi\rangle = \mu|\psi\rangle$

then...

$J_z(J_\pm|\psi\rangle) = (\mu \pm 1) J_\pm|\psi\rangle$

either $\mu \pm 1$ is e-val

or $J_\pm|\psi\rangle = 0$.

$J^2 = J_- J_+ + J_z^2 + J_z$

$\langle\psi| J^2 |\psi\rangle \geq 0$

combine...

max & min value of μ: $\pm\mu_*$

$J^2|\psi_*\rangle = \mu_*(\mu_* + 1)|\psi\rangle \ldots$

all μ's must differ by integer...

$\mu_* \to j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$\boxed{4}$ What is the representation theory of SU(2)?

Started w/ $su(2)$ Lie algebra;

$[J^2, J_z] = 0$: $\quad J^2 |jm\rangle = j(j+1)|jm\rangle$; $\underline{J_z|jm\rangle = m\,|jm\rangle}$

$j = 0, \frac{1}{2}, 1, \ldots \qquad m = -j, 1-j, \ldots, j$

Suppose we have <u>irrep</u> R of $SU(2)$.

$\forall g \in SU(2)$, $\exists R(g)$ obeys $R(g_1)R(g_2) = R(g_1g_2)$ <u>AND</u> R's can't be sim. diag.

$g_1 = 1 + i\delta\theta_1^a T^a \qquad g_2 = 1 + i\delta\theta_2^a T^a$

$$R(g_2^{-1})R(g_1)R(g_2) = (1 - i\delta\theta_2^a R(T^a))(1 + i\delta\theta_1^a R(T^a))(1 + i\delta\theta_2^a R(T^a))$$
$$= 1 + i\delta\theta_1^a R(T^a) - \delta\theta_1^a\delta\theta_2^b [R(T^a), R(T^b)]$$
$$= R(g_2^{-1}g_1g_2) = R(1 + i\delta\theta_1^a T^a - \delta\theta_1^a\delta\theta_2^b[T^a, T^b])$$
$$\Rightarrow [R(T^a), R(T^b)] = R([T^a, T^b]) = f^{abc}R(T^c)$$

in any mom. algebra; $J_\pm|jm\rangle \propto |j(m\pm1)\rangle$: space of fixed j

Can't simultaneously diag. J_x, J_y, J_z ... span $\{|j,-j\rangle, \ldots, |j,j\rangle\}$

Claim: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \ldots$ are all finite-dim irreps is an irrep of $SU(2)$

5 What is the relationship between SU(2) and SO(3)?

Lie algebras $so(3) = su(2)$

up to re-scaling generators

does this mean groups are isomorphic? ~~$SO(3) = SU(2)$~~ no

Why? [deep...] Problem 5d on HW #1:

rotate around z-axis: $SO(3)$: $e^{\theta^z T^z} = \begin{pmatrix} \cos\theta^z & -\sin\theta^z & 0 \\ \sin\theta^z & \cos\theta^z & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$j = \frac{1}{2}$

$SU(2)$: $e^{\theta^z T^z} = e^{-i\frac{1}{2}\sigma^z\theta^z} = \begin{pmatrix} e^{-i\theta^z/2} & 0 \\ 0 & e^{i\theta_z/2} \end{pmatrix}$

$\theta^z = 2\pi$: $SO(3) \to 1$ $SU(2) \to -1$

#1: $j = 0, 1, 2, \ldots$ are irreps of $SO(3)$

#2: spin-$\frac{1}{2}$ electron, etc... predicted by group theory

6 Discuss two common conventions for the irreps of SU(2).

Choice #1 : spin j

irreps : $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

1-dim

2-dim

3-dim

Choice #2: (HEP) dimension

irreps : [boldface] $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \ldots$

1-dim

2-dim

7 What is the representation theory interpretation of the “addition of angular momentum” problem in quantum mechanics?

Example: spin-$\frac{1}{2}$ electron in $\ell=2$ state of hydrogen; what are possible values of total spin?

$$\underset{\ell}{2} \otimes \underset{s}{\tfrac{1}{2}} = \underset{j}{\tfrac{3}{2} \oplus \tfrac{5}{2}}$$

tensor product of irreps

direct sum decomposes into irreps

$|n\ell m\rangle|s\rangle$

Rule: $j_1 \otimes j_2 = |j_1-j_2| \oplus (|j_1-j_2|+1) \oplus \cdots \oplus (j_1+j_2)$

HEP notation: $5 \otimes 2 = 4 \oplus 6$

$\boxed{8}$ What is the representation theory interpretation of the hydrogen atom's selection rules?

H atom [any isotropic system] $\underline{O(3)} = SO(3) \times \mathbb{Z}_2$

$\vec{r} = -\vec{r}$

energy states $|n\ l\ m\rangle$

n: energy level

$\underbrace{l\ m}_{\text{group/rep indices}}$: l irrep $l = 0, 1, 2, \ldots$

$m = -l, 1-l, \ldots, l$

Selection rules: $|n\ l\ m\rangle \to |n'l'm'\rangle$ spont. emission

$l' = l \pm 1$

$m' = m \pm 1$ or m

Why is the $l = l'$ transition forbidden?

$O(3)$ irreps:

$0_+, 1_-, 2_+, 3_-, 4_+, \ldots$

E.g. $l = 2 \to l = 1 \ldots$

$1_- \subset 1_- \otimes 2_+$

$= 1_- \oplus 2_- \oplus 3_-$

$\langle n\,l\,m | \vec{p} | n'l'm'\rangle \neq 0$

$j = 1$

need $l \subset 1 \otimes l'$

$= (l'-1) \oplus l' \oplus (l'+1)$