PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 12

February 23

1 Review the representation theory of SU(2) and SO(3).

algebra corresponds to both SU(2) and SO(3)

$$\left.\begin{aligned}[J_x, J_y] &= iJ_z \\ [J_y, J_z] &= iJ_x \\ [J_z, J_x] &= iJ_y\end{aligned}\right.$$

$$[J^2, J_z] = 0$$

$$|j\,m\rangle$$

$j = 0, \frac{1}{2}, 1, \ldots$

$m = -j, 1-j, \ldots, j$

irreducible represen. of SU(2) of dimension $2j+1$

allowed irreps of SO(3)? $j = 0, 1, 2, \ldots$

in SO(3) we need rotation by 2π to give identity matrix in representation R,

$$R(2\pi \text{ rotation}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

e.g. $j = \frac{1}{2}$: $R(2\pi \text{ rotations}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

NOT representation of SO(3)

2 Interpret $\underbrace{1}\otimes 1 = \underbrace{0\oplus 1\oplus 2}$ in terms of vector "multiplication".

$|j=1, m_1\rangle \otimes |j=1, m_2\rangle$

(uncoupled basis)

$|J\ M\rangle$

(coupled basis)

$J(J+1) = (\vec{J}_1 + \vec{J}_2)^2$

$M = m_1 + m_2$

allowed $J = 0, 1, 2$

"1" is an irrep of SO(3); is 3-dimensional: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

more generally $\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$ "=" A_i $(i \in \{x, y, z\})$

$\vec{A}\otimes\vec{B} \longmapsto A_i B_j \longleftrightarrow \begin{pmatrix} A_x B_x & A_y B_x & A_z B_x \\ A_x B_y & A_y B_y & A_z B_y \\ A_x B_z & A_y B_z & A_z B_z \end{pmatrix} = M$

spin 0?

$\mathrm{tr}(M)$

$A_x B_x + A_y B_y + A_z B_z$

$\vec{A}\cdot\vec{B} = A_i B_i$

spin 1?

$\vec{A}\times\vec{B}$

$\varepsilon_{ijk} A_j B_k$

$M - M^T$

$\mathrm{tr}(R^T M R)$

$= \mathrm{tr}(M R R^T)$

$= \mathrm{tr}(M)$

$\leftarrow$ $M \to R^T M R$

$A_i B_j \to R_{ii'} A_{i'} \times R_{jj'} B_{j'}$

spin 2?

$M + M^T - \frac{2}{3}\mathrm{tr}(M)$

3 What are the representations of O(2)? What about SO(2)?

$$O(2) = \{ M \in \mathbb{R}^{2\times 2} : \underset{\det(M)^2=1}{M^T M = \mathbb{1}} \} \qquad SO(2) = \{ M \in O(2) : \det(M) = 1 \}$$

two one dimensional irreps of O(2): $U_0^+ : U_0^+(M) = 1$

$U_0^- : U_0^-(M) = \det(M)$

two dimensional irreps of O(2): $R_1, R_2, R_3, \ldots$ basis: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \xmapsto{\text{in } R_n} \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix} \to \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix}$$

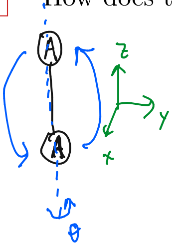
$\cap$ SO(2)

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xmapsto{\text{in } R_n} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

SO(2) irreps: label by integer $n \in \mathbb{Z}$: $M_\theta \mapsto e^{in\theta}$ $\quad S_n(M_\theta) = e^{in\theta}$

$V_0^\pm \to \cancel{}^{(n=0)} S_0 \qquad R_n \to \cancel{n \oplus -n} \quad S_n \oplus S_{-n}$

4 How does the $j = 1$ irrep of SO(3) decompose into irreps of O(2)?



invariant under $O(2) \leq SO(3)$

$$M_\theta = \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$S = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & -1 \end{array}\right)$$

what are electronic states?

Classify orbitals of A atoms into irreps of O(2)!

(s, p, d, ...)

$l = 0, 1, 2, \ldots$

p-state ($l = 1$) (3-dim irrep of SO(3))

$$\left(\begin{array}{c} x \\ y \\ \hline z \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c} x \\ y \end{array}}\right\} R_1 \\ \left.\vphantom{z}\right\} U_0^- \end{array}$$

$$\underset{SO(3)}{1} \longrightarrow \underset{O(2)}{U_0^- \oplus R_1}$$

5 How does the $j=2$ irrep of SO(3) decompose into irreps of O(2)?

$j=2$ irrep of $SO(3) \to$ traceless, symmetric 3×3 matrices

$$N = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} x & y & z \end{array}$$

$$N = \left(\begin{array}{cc|c} -\frac{N_{zz}}{2}+b & a & N_{zx} \\ a & -\frac{N_{zz}}{2}-b & N_{zy} \\ \hline N_{zx} & N_{zy} & N_{zz} \end{array}\right)$$

U_0^+ • N_{zz} alone: $N_{zz} = -N_{xx} - N_{yy}$

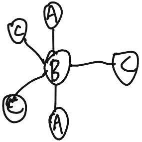
R_1 • $\{N_{zx}, N_{zy}\}$ don't mix w/ other components...

R_2 • traceless symmetric 2×2 matrix

$$\begin{pmatrix} b & a \\ a & -b \end{pmatrix}$$

$$\underset{SO(3)}{j=2} \longrightarrow \underset{O(2)}{U_0^+ \oplus R_1 \oplus R_2}$$

6 Explain why the dihedral groups $D_{2n} \leq O(2)$. How do the irreps of $O(2)$ decompose into irreps of D_{2n}?



e.g. $\rightarrow$ $D_6 \leq SO(3)$

$SO(2)$

$$r = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}\begin{matrix} x \\ y \end{matrix}, \quad s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{matrix} x \\ y \end{matrix}$$

$D_8, D_{12}, \ldots$

Case 1 (n even):

$U_0^+ \leftarrow$

$U_0^- \leftarrow$

$R_1 \leftarrow$ irreps

$R_2 \leftarrow$

$R_3 \leftarrow$

$\vdots$

$R_{n/2} = U^+_{n/2} \oplus U^-_{n/2}$

$\ldots R_{\frac{n}{2}+1} \rightarrow R_{\frac{n}{2}-1}$

$\vdots$

$R_n \rightarrow U_0^+ \oplus U_0^-$

Case 2 (n odd):

U_0^+

U_0^-

R_1

$\vdots$

$R_{\frac{n+1}{2}}$

$R_{\frac{n+3}{2}} \rightarrow R_{\frac{n+1}{2}}$

$\vdots$

7 Re-interpret the irreps of D_8 in terms of branching rules from O(2).

branching rules: $\underset{O(2)}{R_2} \longrightarrow \underset{D_8}{U_2^+ \oplus U_2^-}$

U_0^+, U_0^-, R_1 of O(2) remained irreps of D_8 . . . $R_2 \to U_2^+ \oplus U_2^-$

	U_0^+	U_2^-	U_2^+	U_0^-	R_1
	1	1'	1''	1'''	2
1	1	1	1	1	2
r^2	1	1	1	1	−2
r	1	−1	−1	1	0
s	1	−1	1	−1	0
rs	1	1	−1	−1	0

U_0: $r = 1$ $\qquad$ U_2: $r = -1$

+: don't transform under refl. $\qquad$ ±: transform under s (or not)

R_1: $\begin{pmatrix} x \\ y \end{pmatrix}$ [position]

$R_1 \otimes R_1 = U_0^+ \oplus U_0^- \oplus U_2^+ \oplus U_2^-$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{\text{change of vars}} \begin{pmatrix} \alpha+\gamma & -\beta+\delta \\ \beta+\delta & \alpha-\gamma \end{pmatrix}$

$\alpha : U_0^+$

$\beta : U_0^-$

$U_2^+ : \gamma$

$U_2^- : \delta$

8 Describe the most general form of the elasticity tensor in a two-dimensional crystal with D_{2n} invariance.