PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 13

February 25

1 Review the Lie group SO(N), and its Lie algebra.

$SO(N)$ = group of rotations in N-dimensional space

$= \{ M \in \mathbb{R}^{N\times N} : \det(M) = 1, \ M^T M = 1 \}$

Lie group

Lie algebra: $M = 1 + \varepsilon N + \mathcal{O}(\varepsilon^2)$

$\det(M) = 1 \Rightarrow \operatorname{tr}(N) = 0$

$M^T M = 1 \Rightarrow \boxed{N = -N^T}$ antisymmetric!

a basis for vector space of antisym $\mathbb{R}^{N\times N}$ matrices:

$T_{ij} = -T_{ji} = \vec{e}_i \vec{e}_j^{\,T} - \vec{e}_j \vec{e}_i^{\,T}$

$i, j \in \{1, \ldots, N\}$

$N = \sum \varepsilon_{ij} T_{ij}$

$[T_{ij}, T_{kl}] = \delta_{jk} T_{il} - \delta_{ik} T_{jl} - \delta_{jl} T_{ik} + \delta_{il} T_{jk}$

how many generators? $\binom{N}{2} = \frac{N(N-1)}{2}$

2 What is the transformation of a vector under rotations? What is the transformation of a tensor?

vector: $\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \vec{x}$ ← N (or $= x_i$)

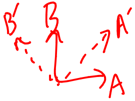
index notation $i \in \{1, \dots, N\}$

N-dim representation of SO(N) = fundamental

transformation of $\vec{x}$: $x_i \longrightarrow M_{ii'} x_{i'}$, $M \in SO(N)$

tensor: $\vec{A} \otimes \vec{B} \longmapsto A_i B_j$

N_{ij}



this representation $N \otimes N$

rank-2 (b/c has 2 indices)

transformation of N_{ij}: $N_{ij} \longrightarrow M_{ii'} M_{jj'} N_{i'j'}$

all indices are subscripts... OK for SO(N)

NOT for SU(N)

3 What is $\mathbf{N} \otimes \mathbf{N}$? this gives reducible representation of $SO(N)$

consider rank-2 tensor X_{ij} ($u_i v_j$: product of 2 "N" irreps)

one irrep: $\mathrm{tr}(X) = X_{ii}$

transforms in the trivial representation $(\mathbf{1})$

another rep: antisymmetric rank-2 tensors: $Y_{ij} = -Y_{ji}$

Proof: $Y_{ij} + Y_{ji} = 0$

$$\xrightarrow{\text{under } SO(N)} M_{ii'} M_{jj'} Y_{i'j'} + M_{jj'} M_{ii'} Y_{j'i'} =$$

$$M_{ii'} M_{jj'} \underbrace{(Y_{i'j'} + Y_{j'i'})}_{=0} = 0$$

[$N=3$: irrep is vector
$N=4$: reducible
$N \geq 5$: irrep (new)]

traceless symmetric: $X_{ij} + X_{ji} - \frac{2}{N} X_{kk} \delta_{ij} = Z_{ij}$

then $Z_{ii} = 0$ $[\delta_{ii} = N]$

irrep

4 In SO(3), the antisymmetic $A_{ij} = -A_{ji}$ irrep is also **3**. Why?

$$-A_{ji} = A_{ij} \longrightarrow \tilde{A}_k \quad ?$$

$(j=1)$

$$\boxed{\varepsilon_{ijk} A_{ij} = \tilde{A}_k}$$

$SO(\cancel{3})$ N invariant tensors?

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\delta_{ij} \to M_{ii'} M_{jj'} \delta_{i'j'}$$
$$= M_{ii'} M_{ji'}$$
$$= M_{ii'} (M^T)_{i'j}$$
$$= (MM^T)_{ij} = \delta_{ij}$$

$$X_{ij}\delta_{ij} \to tr(X)$$
$$= X_{ii}$$

is invariant under $SO(N)$

another invariant:

$$\varepsilon_{ijk} = \begin{cases} +1 & ijk \text{ is permut. of } 123 \text{ w/ sign} = +1 \\ -1 & \text{sign} = -1 \\ 0, & 2 \text{ indices same} \end{cases}$$

why $SO(3)$ invariant?

$$\varepsilon_{ijk} \longrightarrow M_{ii'} M_{jj'} M_{kk'} \varepsilon_{i'j'k'}$$

$$-\varepsilon_{ijk} = \varepsilon_{jik} \longrightarrow M_{ii'} M_{jj'} M_{kk'} \underbrace{\varepsilon_{j'i'k'}}_{= -\varepsilon_{i'j'k'}}$$

$$\varepsilon_{ijk} \xrightarrow{M} E_{ijk}$$

but $E_{ijk} = -E_{jik} = -E_{ikj}$

$$= \cancel{E(M)}\,\varepsilon_{ijk}$$

$$E(M) = \det(M) = 1$$

5 Describe all of the irreps of SO(3) in terms of tensors.

$j=0$ (1) HEP c (scalar)

$j=1$ (3) HEP v_i (vector)

$j=2$ (5) HEP $X_{ij}+X_{ji}-\frac{2}{3}\delta_{ij}X_{kk}$

$3\otimes 3 = 5 \oplus 3 \oplus 1$

$\epsilon_{ijk}X_{ij}$ $\quad \delta_{ij}X_{ij}$

$j=3$ (7) HEP

$(j=2)\otimes(j=1) = (j=3)\oplus(j=2)\oplus(j=1)$

$5 \otimes 3 = 7 \oplus 5 \oplus 3$

A_{ij} B_k

traceless symmetric

$A_{ij}B_j$

$A_{ij}B_k\epsilon_{jkl} + A_{lj}B_k\epsilon_{jki}$

$A_{ij}B_k + A_{jk}B_i + A_{ki}B_j$: fully symmetric

Generalization: spin j irrep (HEP: $2j+1$) correspond to fully symmetric rank j tensor

6 Describe the irreps of $SO(N)$ as tensors.

Scalar : c (1 dim)

fundamental/vector : v_i (N dim)

traceless symmetric rank-2 : $X_{ij} = X_{ji}$, $X_{ii} = 0$ $\left(\frac{N(N-1)}{2} + N - 1 = \frac{(N-1)(N+2)}{2} \text{ dim irrep} \right)$

antisymmetric rank-2 : $X_{ij} = -X_{ji}$ $\frac{N(N-1)}{2}$ dim [new irrep for $N > 4$]

Keep on going : traceless fully symmetric rank 3 : $Y_{ijk} = Y_{jik} = Y_{ikj}$, $Y_{iik} = 0$

(new irrep)

or fully antisym...

or some antisymmetry and symmetry (Young tableaux)

Can use $\varepsilon_{i_1 \ldots i_N}$ to get rid of $\geq \frac{N}{2}$ antisymmetric indices.

7 The non-relativistic hydrogen atom (ignoring electron spin) has a "hidden" symmetry group SO(4). The degeneracy of energy E_n is n^2. Show that irreps exist of this dimension for ~~$n = 1, 2, 3, 4$.~~ all n. (Zee VII.i1)

$m = -\ell, \ldots, +\ell$

$|n \ell m\rangle$

$\ell = 0, \ldots, n-1$

$n = 1, 2, 3, \ldots$

$$H|n\ell m\rangle = E_n |n\ell m\rangle$$

$E_{n\ell}$ expected if SO(3) symmetry

Claim: fully symmetric traceless rank-n tensor, that irrep of SO(4) has dimension $(n+1)^2$

$$T_{ij\cdots n} = T_{ji\cdots n} = T_{nj\cdots i}\,; \quad T_{ii\cdots n} = 0$$

Proof: let D_n be # of fully symmetric rank n tensors.

$$D_n = \underbrace{4}_{T_{11\cdots 1}} + \underbrace{\binom{4}{2}(n-2+1)}_{T_{1 1\cdots 2\cdots 2}} + \underbrace{\binom{4}{3}\frac{(n-1)(n-2)}{2}}_{T_{1\cdots 2\cdots 3\cdots}} + \underbrace{\frac{(n-1)(n-2)(n-3)}{6}}_{T_{1\cdots 2\cdots 3\cdots 4\cdots}}$$

# of traceless/sym tensor

$$C_n = D_n - D_{n-2} = (n+1)^2$$