PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 14

March 2

1 What is the Lie group SU(N)? What are the generators of $\mathfrak{su}(N)$?

Lie group SU(N): $SU(N) = \{ M \in \mathbb{C}^{N\times N};\ \det(M) = 1,\ M^\dagger M = \mathbb{1} \}$

generators: $M = \mathbb{1} + \underset{\text{infinitesimal}}{\varepsilon} N + \mathcal{O}(\varepsilon^2) \longrightarrow \mathbb{1} + i\underset{\text{infinitesimal}}{\theta} N + \cdots$

$$\det(M) = 1 = \det(\mathbb{1} + \varepsilon N + \cdots) = 1 + \varepsilon\, tr(N) + \cdots \Rightarrow tr(N) = 0$$

$$(\mathbb{1} + \bar{\varepsilon} N^\dagger)(\mathbb{1} + \varepsilon N) = \mathbb{1} + \bar{\varepsilon} N^\dagger + \varepsilon N = \mathbb{1} \Rightarrow \varepsilon = i\underset{\text{real}}{\theta},\ N = N^\dagger$$

how many generators?

$$(N^\dagger)_{ij} = \bar{N}_{ji}$$

- diagonals; $N_{11}, N_{22} \in \mathbb{R}$ AND $N_{kk} = 0 = tr(N)$
  $N-1$ generators that are diagonal
- off-diagonals;

$$\begin{pmatrix} 0 & a+bi & \\ a-bi & 0 & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \ddots \end{pmatrix} a + \begin{pmatrix} 0 & i & \\ -i & 0 & \\ & & \ddots \end{pmatrix} b$$

$$\underset{\text{a\&b}}{2} \times \underset{\text{pairs i\&j}}{\binom{N}{2}} = N(N-1)$$

total # of generators: $N^2 - 1$

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Introduce an upper/lower index notation for handling tensor transformations under SU(N).

Recall: SO(N) $\quad M \in SO(N) \quad \underbrace{M^T M = 1}$ $\rightarrow$ $\boxed{M_{ki}M_{kj} = M_{ik}M_{jk} = \delta_{ij}}$

built irreps out of tensors...

$$T_{kk\cdots} \xmapsto{M} \underbrace{M_{ki}M_{kj}}_{\delta_{ij}} T_{ij\cdots} = T_{kk\cdots}$$

What doesn't work in SO(N): $\quad M_{ik}M_{kj} \neq \delta_{ij}$

$$= (M^2)_{ij}$$

$U \in SU(N)$

Turn to SU(N): $\quad U^\dagger U = 1 = UU^\dagger$

$$(U^\dagger)_{ik}U_{kj} = \delta_{ij} = \bar{U}_{ki}U_{kj}$$

$$U_{ik}(U^\dagger)_{kj} = U_{ik}\bar{U}_{jk} = \delta_{ij}$$

$$U_{ij} \longrightarrow U^i{}_j \qquad \bar{U}_{ij} = \bar{U}_i{}^j$$

$$\bar{U}_k{}^i U^k{}_j = \delta^i_j = U^i{}_k \bar{U}_j{}^k$$

ordinary vector: $\quad x^i \quad [\vec{x} \to U\vec{x}]$

$$x^i \longmapsto U^i{}_j x^j$$

[only contract raised index w/ lowered]

another vector:

$$y_i \rightarrow \bar{U}_i{}^j y_j \qquad [\vec{y} \to U^\dagger \vec{y}]$$

3 How do we write irreducible representations of SU(N) in terms of tensors?

scalar (trivial) 1-dim : $\mathbb{C} \xmapsto{U\in SU(N)} \mathbb{C}$ $\quad U \mapsto 1$ (bold)

fundamental (N-dim) (vector): $x^i \longrightarrow U^i{}_j\, x^j$ $\quad (N)$

anti-fundamental (conjugate vector): $y_i \to \bar{U}_i{}^j\, x_j$ $\quad (\bar{N})$

distinct irreps [when N > 2]

proceed a la SO(N):

$N \otimes \bar{N}$: $X^i{}_j$ $\quad$ $X^i{}_j\, \delta^j{}_i = tr(X)$ in $\mathbf{1}$

one invariant tensor is δ^i_j

$$N \otimes \bar{N} = 1 \oplus (N^2 - 1)$$

$X^i{}_j$ $\quad$ $tr(X)$ $\quad$ $X - \frac{1}{N} tr(X)$

[traceless)

$N \otimes N$: Y^{ij} → $Y^{ij} + Y^{ji}$ symmetric

→ $Y^{ij} - Y^{ji}$ antisymmetric

DOES NOT have 1 [for N > 2]

4 How do we use the Levi-Civita tensor to raise and lower indices?

$$\det(\underset{\nwarrow \in SU(N)}{U}) = 1$$

$$U^{i_1}{}_{j_1} U^{i_2}{}_{j_2} \cdots U^{i_N}{}_{j_N} \varepsilon^{j_1 \cdots j_N} = \varepsilon^{i_1 i_2 \cdots i_N}$$

$$\downarrow$$

$$\varepsilon^{12\cdots N} = 1 \qquad \varepsilon^{11\cdots} = 0$$

$$\varepsilon^{2134\cdots N} = -1$$

$$U^{i_1}{}_{j_1} \cdots U^{i_N}{}_{j_N} \varepsilon_{i_1 \cdots i_N} = \varepsilon_{j_1 \cdots j_N}$$

similar to $SO(N)$: if e.g. tensor N raised antisymmetric indices

$$A^{i_1 \cdots i_N} = -A^{i_N \cdots i_1}$$

$$\hookrightarrow \varepsilon_{i_1 i_2 \cdots i_N} A^{i_1 \cdots i_N} = \tilde{A}$$ in the trivial

$$A^{i_1 \cdots i_{N-1}} = -A^{i_2 i_1 i_3 \cdots i_{N-1}} \qquad \varepsilon_{i_1 \cdots i_N} A^{i_1 \cdots i_{N-1}} = \tilde{A}_{i_N}$$

5 Express the irreps of SU(2) in terms of tensors.

Recall irreps of SU(2): $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

HEP : $1, 2, 3, 4, 5 \ldots$

(scalar) c

(vector) x^i

antisym: $X_A^{ij} = -X_A^{ji}$

$$X^{ij} = \overset{\text{sym}}{X_S^{ij}} + \overset{\text{antisym}}{X_A^{ij}}$$

$\varepsilon_{ij} X^{ij} = \varepsilon_{ij} X_A^{ij}$ is scalar!

general X^{ij} is $2 \otimes 2 = \underset{S}{3} \oplus \underset{A}{1}$

Why is no $\bar{2}$?

$$y_i \mapsto \varepsilon^{ij} y_j = \tilde{y}^i$$

In general: use ε^{ij} to raise any lowered index in any proposed irrep $T^{i\cdots}{}_{j\cdots}$

Claim: q-dimensional irrep of SU(2) correspond to fully symmetric tensor w/ q−1 raised indices.

6 Express the irreps of SU(3) in terms of tensors.

Claim: All irreps of SU(3) are tensors w/ m fully symmetric raised, n fully symmetric lowered indices: (m, n)

$$\varphi^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} = \varphi^{i_2 i_1 i_3 \cdots i_m}{}_{j_1 \cdots j_n} = \varphi^{i_1 \cdots i_m}{}_{j_2 j_1 j_3 \cdots j_n} \leftarrow \underline{\underline{\text{traceless}}}$$

Why?: Suppose $\varphi^{ik\cdots}{}_{\cdots} = -\varphi^{ki\cdots}{}_{\cdots}$

$$\tilde{\varphi}_j{}^{\cdots}{}_{\cdots} = \varepsilon_{ikj}\varphi^{ik\cdots}{}_{\cdots}$$

$$\varphi^k{}_j = \underset{\substack{(0,0)\\(\text{trivial})}}{\varphi^m{}_m \frac{1}{3}\delta^k{}_j} + \underset{(1,1)}{[\text{traceless piece}]}$$

7 How do we multiply together irreps of SU(3)?

$(m,n)\otimes(m',n') = ?$

$$\varphi^{(i_1\cdots \overbrace{i_m)(k_1}^{\varepsilon}\cdots k_{m'})}_{(j_1\cdots \underbrace{j_n)(l_1}_{\varepsilon?}\cdots l_{n'})}$$

φ^{ik}_{jl}

#1: contract i w/ l or k w/ j.

#2: ik might be antisym... or jl

Example: $(1,1)\otimes(1,1) = (0,0)\oplus(1,1)\oplus(1,1)\oplus(0,3)\oplus(3,0)\oplus(2,2)$

$$\varphi^{ij}_{ji} \oplus \varphi^{ij}_{jl} \oplus \varphi^{ik}_{ji} \oplus \varphi^{ik}_{jl}$$

(0,0) traceless (1,1) (1,1) (1,1; 1,1)

$\varepsilon_{ikm}\varphi^{ik}_{jl} = \varphi_{jlm}$ [this symmetric] (0,3)

$\varepsilon^{jln}\varphi^{ik}_{jl} = \varphi^{ikn}$ (3,0)

or φ^{ij}_{kl} already sym. (2,2)

8 What are the dimensions of the (m, n) irrep of SU(3)?

Let D_{mn} = # of fully sym tensors w/ m raised, n lowered indices

C_{mn} = # of . . . traceless

$$D_{mn} = \underset{\text{raised indices only}}{d_m}\ \underset{\text{lowered indices only}}{d_n}$$

$$= \frac{1}{4}(n+1)(n+2)(m+1)(m+2)$$

$$d_n = \underset{T^{11\cdots1}}{3} + \underset{T^{11\cdots2\cdots}}{\overset{3}{\overset{\shortparallel}{\binom{3}{2}}}(n-1)} + \underset{T^{1\cdots2\cdots3}}{\frac{(n-1)(n-2)}{2}}$$

$$C_{mn} = D_{mn} - D_{m-1,n-1} = \frac{1}{2}(n+m+2)(m+1)(n+1)$$

$$(1,1)\otimes(1,1) = (0,0)\oplus(1,1)\oplus(1,1)\oplus(0,3)\oplus(3,0)\oplus(2,2)$$

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus \overline{10} \oplus 10 \oplus 27$$