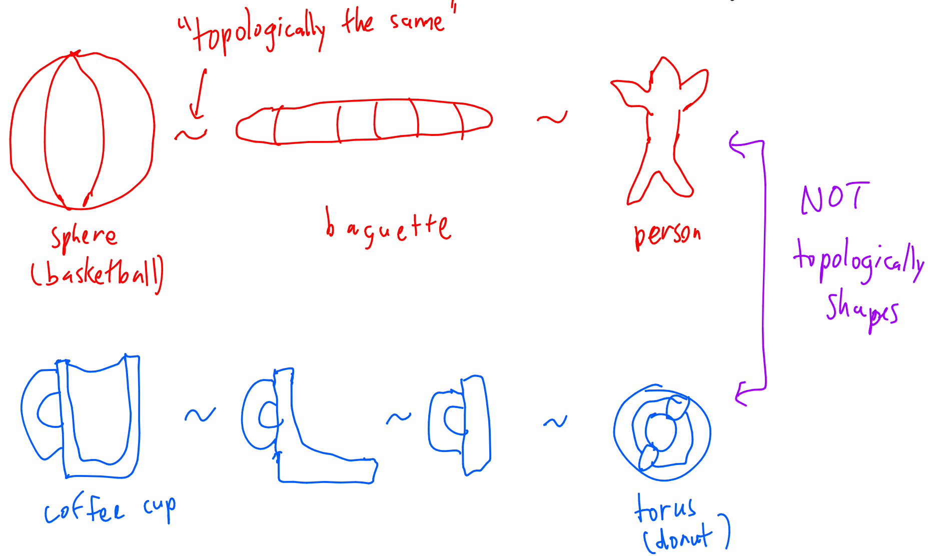
PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 15

March 4

1 In intuitive terms, what is the mathematical purpose for topology?

topology = study of classifying shapes under any deformation (except cut / glue)



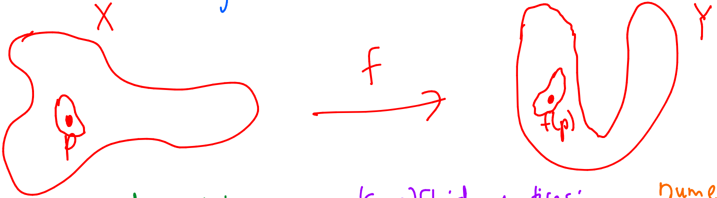
2 Define a homeomorphism; give examples of topological spaces that show up in physics.

topologically equivalent = homeomorphic

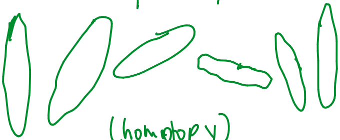
sets X and Y are homeomorphic if there's an invertible $\boxed{\text{continuous}}$ function $f: X \to Y$.

Math question: What structure do X & Y need for "continuous" to exist (see Nakahara 2.3)

... ignore this question.

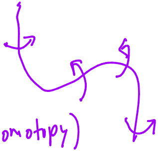


nematic liquid crystals:



(homotopy)

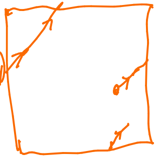
(Super)Fluid vortices:



(homotopy)

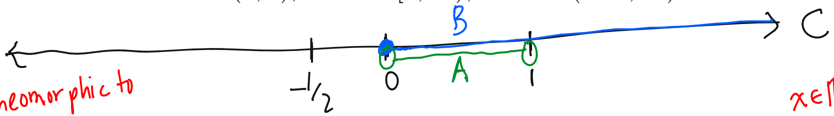
numerics: periodic BCs

(homology)



3 Which of these spaces are homeomorphic?

$$A = (0,1), \qquad B = [0,\infty), \qquad C = (-\infty,\infty) = \mathbb{R}$$



homeomorphic to

$A \cong C$ $\quad f: C \to A$ is $f(x) = \dfrac{e^x}{1+e^x}$ $\quad$ since $0 < e^x < \infty$, $\quad x \in \mathbb{R}$

$0 < e^x < 1 + e^x$, so $0 < f < 1$

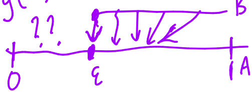
"boundedness" is NOT topological invariant

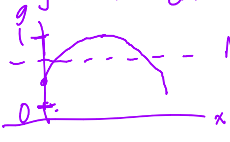
construct inverse: $\quad y = \dfrac{e^x + 1 - 1}{1+e^x} = 1 - \dfrac{1}{1+e^x}$

$\ldots \quad x = \log\left[\dfrac{1}{1-y} - 1\right]$

$A \not\cong B$: if homeomorphism did exist... $g: B \to A$ continuous/invertible...

$g(0) = \varepsilon \in A$

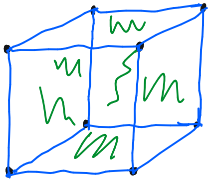




NOT invertible.

4 Define the Euler characteristic for a polyhedron in two dimensions.

vertices $V = 8$ edges $E = 12$ faces $F = 6$

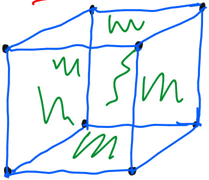


$$\chi = V - E + F = 2$$

Euler characteristic

5 Calculate the Euler characteristic for a cube and a tetrahedron.

cube



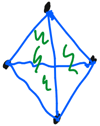
$V = 8$

$E = 12$

$F = 6$

$\chi = 8 - 12 + 6 = 2$

tetrahedron



$V = 4$

$E = 6$

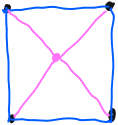
$F = 4$

$\chi = 4 - 6 + 4 = 2$

sphere

$\chi = 2$

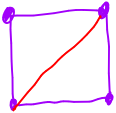
6 Why do all shapes "topologically equivalent" to a sphere have the same Euler characteristic?



$$\Delta V = 1 - 0 = 1$$
$$\Delta E = 4 - 0 = 4$$
$$\Delta F = 4 - 1 = 3$$

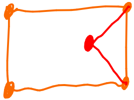
$$\Delta \chi = \Delta V - \Delta E + \Delta F = 0$$

add edge:



$$\Delta V = 0$$
$$\Delta E = 1$$
$$\Delta F = 1$$
$$\Delta \chi = 0$$

add vertices



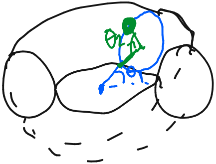
$$\Delta V = 1$$
$$\Delta E = 2$$
$$\Delta F = 1$$
$$\Delta \chi = 0$$

remove things is just reversing these steps (or combinations of)

$\Downarrow$

$\chi = 2$ is a topological invariant

7 Calculate the Euler characteristic of the torus.

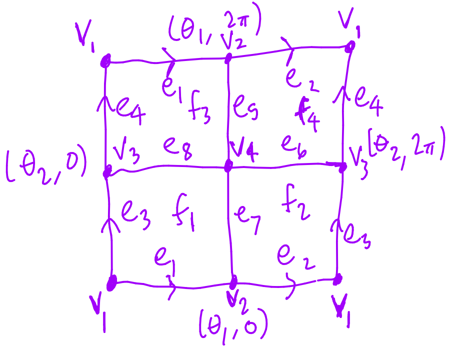


$(\theta_1, \theta_2) = \bigcirc \times \bigcirc$

$\theta_{1,2} \sim \theta_{1,2} + 2\pi$

define equivalent

Recall:



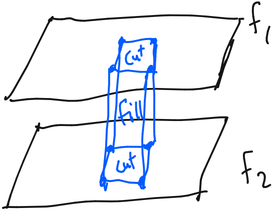
$V = 4$

$E = 8$

$F = 4$

$\boxed{\chi = 0}$

8 What is the Euler characteristic of a two-dimensional surface of genus g?



Sphere: $\chi = 2$

$\downarrow$

torus: $\chi = 0$

"surgical operation"

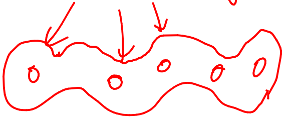
$\Delta V = 8$

$\Delta E = 12$

$\Delta F = 4 - 2 = 2$

$\Delta \chi = -2$

g "holes": genus g surface



$\chi = 2 - 2g$