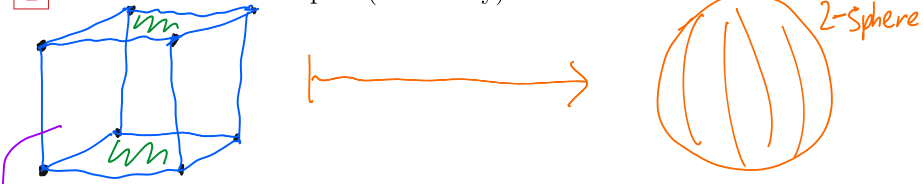
PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 16

March 9

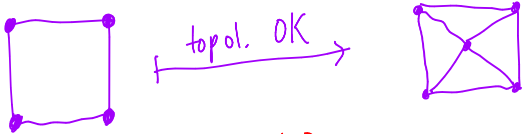
1 What is a CW-complex (informally)?



8 vertices – 0-dim skeleton
12 edges – 1-dim skeleton
6 faces – 2-dim object

$$\chi = V - E + F$$

face



$$\left.\begin{aligned}\Delta V &= 1\\ \Delta E &= 4\\ \Delta F &= 3 = 4-1\end{aligned}\right\}\Delta\chi = 0$$

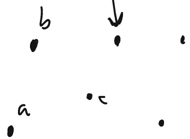
CW-complex (intuitively)

2 What is a CW-complex (formally)?

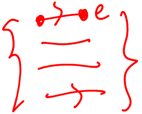
recursive construction:

0 - skeleton: collection of points
(dimension)

0 - cell

$X^0 =$ 

$= \{a, b, c \ldots\}$

$X^1 = X^0 \cup$ 

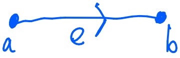
Define equivalence relation

$\partial e = \{a_0, b_0\}$

$a_0 \sim a$

$b_0 \sim b$

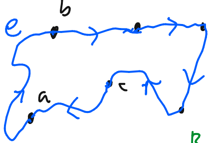
1-cell: line segment



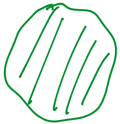
$= D^1$ (1-dim disk)

boundary

$S^0 = \partial D^1 = \{a, b\} \sim$ "$b - a$"

$X^1 =$ 

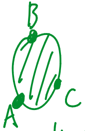
2-cell:

$D^2 =$ 

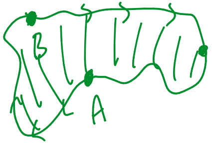
boundary:

$\partial D^2 = S^1$

1-dim sphere

$\hat{X}^2 = X^1 \cup$ 

equiv relation glues disk boundary to X^1

$X^2 =$  $= \hat{X}^2 / \sim$

3 Give 2 CW-complexes for the n-sphere, S^n.

algebraic definition:

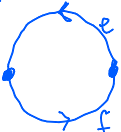
$$D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \dots + x_n^2 \le 1\}$$

$n = 1, 2, \dots$

$$S^n = \partial D^{n+1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_{n+1}^2 = 1\}$$

#1 (recursive): $S^0 \subset S^1 \subset \dots \subset S^n$

$S^0 =$ $\underset{-1}{\cdot}$ $\underset{+1}{\cdot}$ $\qquad x_1^2 + \underbrace{0^2 + \dots + 0^2}_{n \text{ times}} = 1$

$S^1 =$ 

$e = (x_1, x_2, 0, \dots 0)$, $x_2 \ge 0$

$f = (x_1, x_2, 0, \dots 0)$, $x_2 \le 0$

$S^2 =$ 

$\cup$ (2-cell, $x_3 \ge 0$)

V (2-cell, $x_3 \le 0$)

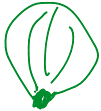
etc.

S^n: glue 2 n-cells onto S^{n-1}

#2: $\{\underset{v}{\cdot} \cup D^n\}/\sim$

(0-cell) (n-cell)

if $x \in \partial D^n$, $x \sim v$

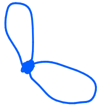


4 Give a CW-complex for the 2-torus $T^2 = S^1 \times S^1$.

0 - skeleton: v

1 - skeleton: $e \cup f$

$\partial e \sim v$ $\partial f \sim v$



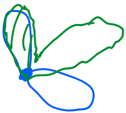
2-cell U: how do we attach??

∂U is attached along the trajectory

$f \to e \to f^{-1} \to e^{-1}$

(reverse)

glue: (f)



glue (fe)

5 More generally, describe a product space $X \times Y$. How do we put a CW complex on it?

n dim m dim

$X \times Y$ = CW-complex of n+m dimensions

CW complex

$(x, y) \in (X, Y)$

built out of e_X^α

CW-complex $X \times Y$ will be built out of cells "$e_X^\alpha \times e_Y^a$"

built out of e_Y^a

E.g. torus (2 dimension) $T^2 = S^1 \times S^1$

$(x, y) \sim (x+1, y) \sim (x, y+1)$

$\in \mathbb{R}^2$

0-cell

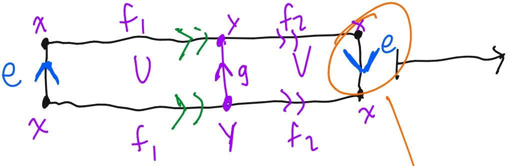
edge

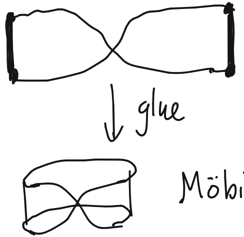
$S^1 = V \cup e$ $\quad \partial e \sim V$

$\partial (V_1 e_2) \sim V_1 (\partial e_2) \sim V_1 V_2$

0+0	0+1		
$V_1 V_2$	$V_1 e_2$	$e_1 V_2$	$e_1 e_2$
0-cell	1-cell	1-cell	2-cell

What is the Klein bottle?





Möbius strip

Euler characteristic:

$V = 2$

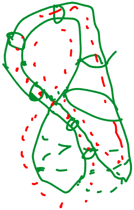
$E = 4$

$F = 2$

$\chi = 0$

flip orientation to get torus

Klein bottle:



χ is NOT sufficient to classify 2d shapes

BUT $\{\chi$ & orientable$\}$ is sufficient.