PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 17

March 11

1 Sketch the low energy theory of a superfluid. Why can topology be relevant?

QM: many bosons (N) in the same quantum state

very low energy: $\Psi(x_1, \ldots, x_N) = \prod_{i=1}^{N} \psi(x_i)$

ψ effective single-particles

[Hartree-Fock approx]

$(\hbar = 1)$

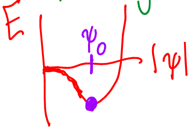
interactions... $g > 0$

$$i\partial_t \psi = -\frac{1}{2m}\nabla^2\psi + \cancel{V_{s.p.}(x)}\,\psi + g|\psi|^2\psi$$

chemical potential $\rightarrow \mu$

$\mu < 0$; superfluid phase

$$E[\psi] = \int d^d x \left[\cancel{\frac{1}{2m}|\nabla\psi|^2} + \frac{1}{2}\mu|\psi|^2 + \frac{g}{4}|\psi|^4\right]$$



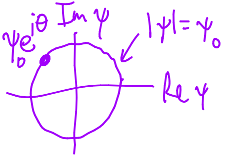
low energy physics:

$\psi(x) = \psi_0 e^{i\theta(x)}$

$\theta \in \mathbb{R}$

$$E_{IR} \approx \int d^d x \, \frac{\psi_0^2}{2m} |\nabla\theta|^2$$

But $\theta(x) \sim \theta(x) + 2\pi$



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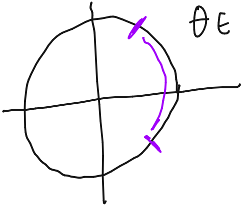
What is a manifold?



Low energy of superfluid:

function $\theta : M \to S^1$

space

Well-defined to topologist...

Physicist: $E = \frac{\rho_0^2}{2m} \int_M d^d x \underbrace{|\nabla\theta|^2}_{???}$

Formal: n-dimensional manifold M is union of (open) sets $\{U_1, \ldots, U_N\}$, such that

- each open set is homeomorphic to a subset of $\mathbb{R}^n$
- $f_i : U_i \to \mathbb{R}^n$

in $U_i \cap U_j$ $\quad \mathbb{R}^n \xrightarrow{f_j^{-1}} M \xrightarrow{f_i} \mathbb{R}^n$

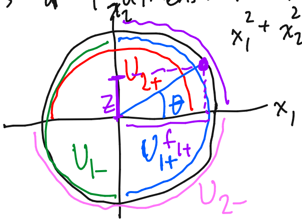
$f_i(f_j^{-1}(x))$ must be smooth (∞-ly many derivatives)

Math: S^1 is a manifold.

Intuitively: manifold = "space where derivatives exist"

3 What is an atlas for the sphere S^n?

S^1 is a 1-dimensional manifold



$U_{1+} = \{(x_1, x_2) \in S^1 : x_1 > 0\}$

$f_{1+} : U_{1+} \to (-1, 1)$
$= x_2$

$f_{2+} : U_{2+} \to (-1, 1)$
$= x_1$

$U_{1+} \cap U_{2+} = \{(x_1, x_2) \in S^1 : x_1, x_2 > 0\}$

$f_{2+}(f_{1+}^{-1}(z)) = \sqrt{1 - z^2}$

$\{U_{1+}, U_{2+}, U_{1-}, U_{2-}\}$ = atlas

(U_{1+}, f_{1+}) = chart

generalize to $S^n = \{(x_1, \dots, x_{n+1}) : x_1^2 + \dots + x_{n+1}^2 = 1\}$

$U_{1+} = \{\vec{x} \in S^n : x_1 > 0\}$

$f_{1+} : U_{1+} \to (-1, 1)^n$
$= (x_2, x_3, \dots, x_{n+1})$

atlas w/ $2n+2$ charts

(∞ derivatives)

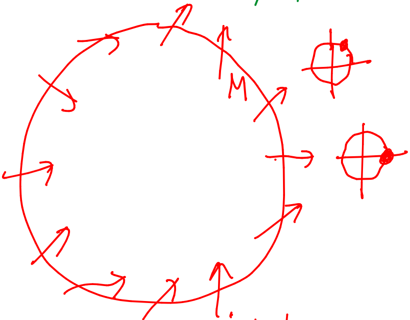
4 What are the 2 ways that manifolds can show up in physics?

Recall: superfluid DOF $\theta \in S^1$

spatial domain may also be a [non-trivial] manifold M

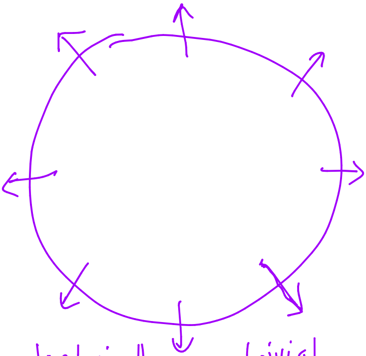
Sneak peek:

global topological difference btwn S^1 and $\mathbb{R}$ responsible



topologically trivial

→ relax to equilibrium



topologically non-trivial

(vortex)

5 What is an n-dimensional cycle on a manifold M? When is it contractible?

n-cycle = smooth map from $S^n \to M$

(n = dimensional)

Example #1: ($n=1$)

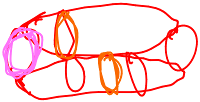
$\mathbb{R}^2 = M$

contractible

(not topologically interested)

Example #2: identity $S^n \to S^n$

Example #3:



NOT contractible

6 Formally speaking, what are differentiable functions on a manifold?

SF example:

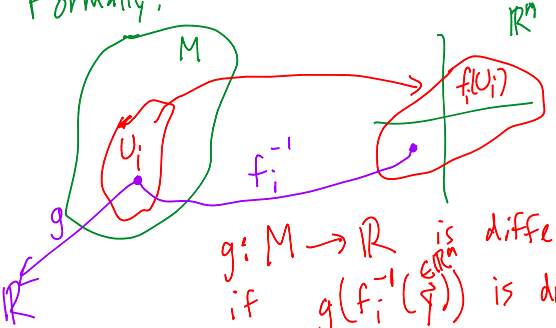
$$E_{IR} = \frac{\psi_0^2}{2m} \int_{\text{space}} d^d x \, (\nabla\theta)^2 \quad \hookrightarrow \theta \in S^1$$

Intuitive answer:

$$\theta(x) \sim \theta(x) + 2\pi$$

$$\nabla\theta \sim \nabla(\theta + 2\pi) = \nabla\theta + 0$$

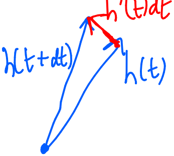
Formally:



$g: M \to \mathbb{R}$ is differentiable if $g(f_i^{-1}(\underset{\in \mathbb{R}^n}{y}))$ is differentiable

[Charts should be compatible: unique answer in $U_i \cap U_j$]

7 Explain why the derivative on a manifold is a vector.

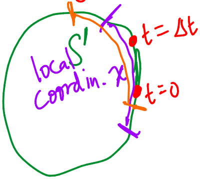


Recall: curve in $\mathbb{R}^n$ is $\vec{h}: \underset{(t)}{\mathbb{R}} \to \mathbb{R}^n$

Able to define $\dfrac{d}{dt}\vec{h} = \left(\dfrac{dh_1}{dt}, \dfrac{dh_2}{dt}, \dots, \dfrac{dh_n}{dt}\right)$

What about manifolds?

$g: \mathbb{R} \to S^1$



as $\Delta t \to 0$, derivative $\to \infty$?

$\dfrac{dg}{dt}$ does <u>NOT</u> exist in S^1

Local coordinate chart:

$q: S^1 \to \mathbb{R}$

$q(g(t)): \mathbb{R} \to \mathbb{R}$

$\dfrac{dq}{dt} =$ "$\dfrac{\partial q}{\partial g}\dfrac{\partial g}{\partial t}$"

globally, does NOT exist

but locally can make sense of it w/ charts:

$= \underbrace{\left(\dfrac{dx}{dt}\dfrac{\partial}{\partial x}\right)}_{\text{vector}} q$

not depending on chart