# PHYS 5040

# Algebra and Topology in Physics

# Spring 2021

# Lecture 18

March 16

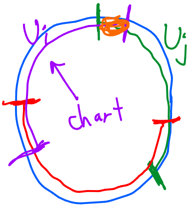
**1** Review the notion of manifolds, and derivatives on manifolds.

manifold = "smooth space where derivatives exist"

Since we have calculus on functions from $\mathbb{R}^n \to \mathbb{R}^n$, define $n$-dimensional manifold analogously

e.g. circle $S^1$ (1-dimensional sphere)

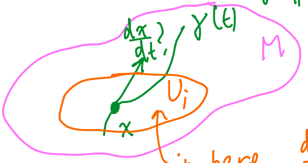
$\{(x, y) : x^2 + y^2 = 1\}$




in each chart

$f_i : U_i \longrightarrow \mathbb{R}$ ← $\mathbb{R}^n$

$\gamma : \mathbb{R} \to M$




local coordinate

in here, $\frac{dx^\mu}{dt}$

auxiliary $g : M \to \mathbb{R}$

$\frac{d}{dt}\left[g(\gamma(t))\right]$ defined everywhere

$= \left[\frac{dx^\mu}{dt}\frac{\partial}{\partial x^\mu}\right] g$ → independent of $g$; right way to think about vector

$f_i(f_j^{-1}(x))$ is $\mathbb{R} \to \mathbb{R}$ must be smooth ($\infty$'ly differentiable)

## 2 What is a 1-form?

Recall: vectors → derivatives

$$\mathbf{V}(\overset{x \in M}{x}) = V^\mu(x) \frac{\partial}{\partial x^\mu}$$

in local coordinates

Mathematician: dual space of vectors [1-forms]

$\omega$

$(\text{1-form})_x : \{\text{vector fields}\}_x \longrightarrow \mathbb{R}$

1) $\omega(V_1 + V_2) = \omega(V_1) + \omega(V_2)$

2) $\omega(\underset{\in \mathbb{R}}{\alpha} V) = \alpha\, \omega(V)$

3) if $\omega(V) = 0$, then $V = 0$ $[V^\mu = 0]$

- $\omega$'s elements of [at each $x$] an $n$-dimensional "vector space"

$$\omega = \omega_\mu \, dx^\mu$$

$n$ basis 1-form

- Choose this basis

$$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$$

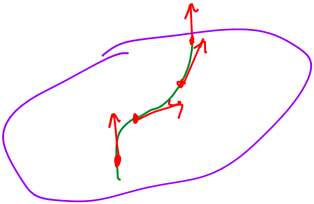
$$dx^\mu(V) = V^\mu(x)$$

**3** Explain how to integrate a 1-form along a curve.

"Velocity" on manifolds $\left[\partial/\partial x^\mu\right]$: instantaneous changes
how much accumulation $\left[\int\right]$ over time?

$$\text{"}\int \vec{V}\cdot d\vec{s}\,\text{"} \rightarrow \sum \Delta t\, \underbrace{\frac{d\vec{s}}{dt}\cdot\vec{V}}_{\mathbb{R}} \rightarrow \sum \Delta t\ \omega(V)$$

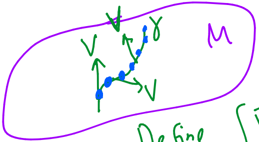
$$\rightarrow \int dt\ \omega(V(t))$$



$$\rightarrow \int \omega(dt\, V(t)) \rightarrow \int_\gamma \omega$$

$$\int_\gamma \omega = \int dt\ \omega\left(\frac{d\gamma}{dt}\right) = \int\left(\frac{dt}{d\tau}d\tau\right)\omega\left(\frac{d\gamma}{dt}\frac{d\tau}{dt}\right)$$

$$t \rightarrow t(\tau) \qquad = \int \underbrace{\cancel{\frac{d\tau}{dt}\frac{dt}{d\tau}}}_{\rightarrow 1}\, d\tau\, \omega\left(\frac{d\gamma}{d\tau}\right)$$

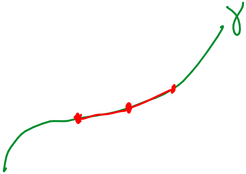



$\omega = \omega_\mu dx^\mu$. Why?

$$\int_\gamma \omega = \int_\gamma \omega_\mu dx^\mu$$

Define $\int \vec{V}\cdot d\vec{s}$?

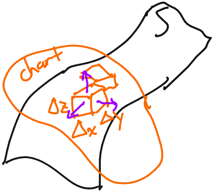
## 4 What is a 2-form?




$$\sum \omega_x \Delta x + \omega_y \Delta y \longrightarrow \int_\gamma \omega$$

$dx \wedge dx = 0.$

$d\vec{a}_i \sim \frac{1}{2}\epsilon_{ijk}\, dx^j \wedge dx^k$


Is there an object (2-form) $\omega_2$ such that $\int_S \omega_2$ exists?

$$\int \vec{w} \cdot d\vec{a} = \sum_{\text{faces}} \overset{\vec{w}_y}{\omega_{xz}} \overbrace{\Delta x \Delta z}^{da_y} + \omega_{yz} \Delta y \Delta z + \omega_{\cancel{z}yx} \Delta \cancel{z} y \Delta x$$

$$\int_S \omega_2 = \int_S \Big[ \omega_{xz} \underbrace{dx \wedge dz}_{\text{area element on face in } x \,\&\, z \text{ directions}} + \omega_{yz}\, dy \wedge dz + \omega_{yx}\, dy \wedge dx \Big]$$

AND orientation: $dx \wedge dz = - dz \wedge dx$

## 5 What is an $r$-form?

On $n$-dim manifold $(n \geq r)$, define $r$-form $w_r$

$$w_r = \frac{1}{r!} \sum_{i_1,\dots,i_r=1}^{n} w_{i_1 \cdots i_r} \underbrace{dx^{i_1} \wedge dx^{i_2} \wedge \cdots dx^{i_r}}_{\binom{n}{r} \text{ basis } r\text{-forms}}$$

FULLY ANTISYMMETRIC: $w_{i_1 i_2 \cdots i_r} = - w_{i_2 i_1 \cdots i_r}$

$$dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_r} = - dx^{i_2} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

$$\underset{r\text{-dim faces}}{\sum} w_{i_1 \cdots i_r} \Delta x_{i_1} \cdots \Delta x_{i_r} \longrightarrow \underset{\substack{r\text{-dim} \\ \text{surface}}}{\int} w_r$$

## $\boxed{6}$ What is the wedge product?

$p$-form $\omega$ & $q$-form $\eta$

$$\omega \wedge \eta = \frac{1}{p!}\frac{1}{q!}\sum_{i_1,\dots,i_p=1}^{n}\sum_{j_1,\dots,j_q=1}^{n} \omega_{i_1\dots i_p}\,\eta_{j_1\dots j_q}\, dx^{i_1}\wedge\dots\wedge dx^{i_p}\wedge dx^{j_1}\wedge \dots \wedge dx^{j_q}$$

Example: 2 dimensions, $p=q=1$

$$\omega = y^2 dx + dy \qquad \eta = x^3 dx + y^3 dy$$

$dy \wedge dy = 0$

$$\omega \wedge \eta = [y^2 dx + dy] \wedge [x^3 dx + y^3 dy]$$

$$= \cancel{y^2 x^3 dx \wedge dx} + x^3 dy \wedge dx + y^5 dx \wedge dy + \cancel{y^3 dy \wedge dy}$$

$$= (x^3 - y^5)\, dy \wedge dx$$

**Proposition:**

1) $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$

2) $(\omega \wedge \eta) \wedge \psi = \omega \wedge (\eta \wedge \psi)$

**7** What is the Hodge dual operation?
(Hodge star)

fully antisymmetric

Recall if $p$-form $\omega$, $\omega = \frac{1}{p!} \omega_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$

$\omega \longmapsto$ "equivalent" rank $n-p$ differential form

$$\tilde{\omega}_{i_1 \cdots i_{n-p}} = \frac{1}{p!} \varepsilon_{i_1 \cdots i_n} \omega_{i_{n-p+1} \cdots i_n}$$

$$* dx^{i_1} \wedge \cdots \wedge dx^{i_p} = \frac{1}{(n-p)!} \varepsilon_{i_1 \cdots i_n} dx^{i_{p+1}} \wedge \cdots \wedge dx^{i_n}$$

Hodge star: $\tilde{\omega} = * \omega$

(loosely $** \omega = \pm \omega$)

**8** Discuss electromagnetism in the language of differential forms. Define magnetic flux using only differential forms and integrals.