PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 19

March 18

1 Review what a differential form is.

Suppose we have n-dimensional manifold. Then an r-form $(r \le n)$ is fully antisymmetric rank-r tensor

$$\omega = \frac{1}{r!} \omega_{i_1 \cdots i_r} \underbrace{dx^{i_1} \wedge \cdots \wedge dx^{i_r}}_{\text{basis } r\text{-form}} \qquad dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_r} = - dx^{i_2} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

Natural integral of ω over an r-dim surface S; ← needs orientation

$\int_S \omega$, independent of coordinates/atlas...

wedge product:

$$\omega_r \wedge \eta_q = \frac{1}{r!} \frac{1}{q!} \omega_{i_1 \cdots i_r} \eta_{j_1 \cdots j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}$$

Hodge star/dual: p-form $\longrightarrow$ $(n-p)$-form

$$* \, dx^{i_1} \wedge \cdots \wedge dx^{i_p} = \frac{1}{(n-p)!} \varepsilon_{i_1 \cdots i_n} dx^{i_{p+1}} \wedge \cdots \wedge dx^{i_n}$$

2 What is the exterior derivative?

$$d\,\omega_r = dx^j \wedge \frac{1}{r!}\frac{\partial \omega_{i_1\dots i_r}}{\partial x^j}\, dx^{i_1}\wedge\dots\wedge dx^{i_r}$$

exterior derivative

r-form

ω r form $\rightarrow$ $d\omega$ $(r+1)$ form

Example: 3-dimensions, 1-form: $\omega = z\,dx + y\,dy + xz\,dz$

$$d\omega = 1\cdot dz\wedge dx + \cancel{dy\wedge dy} + z\,dx\wedge dz + \cancel{x\,dz\wedge dz}$$
$$= (z-1)\,dx\wedge dz$$

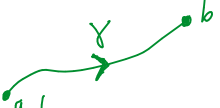
Proposition: $d(d\omega) = d^2\omega = 0.$

Proof: $$d(d\omega) = \underbrace{dx^k\wedge dx^j}_{jk \text{ antisymmetric}}\wedge \frac{1}{r!}\underbrace{\frac{\partial^2 \omega_{i_1\dots i_r}}{\partial x^j \partial x^k}}_{jk \text{ symmetric}}\, dx^{i_1}\wedge\dots\wedge dx^{i_r}$$
$$= 0$$

"0-form": scalar function

$$df = (\partial_i f)\, dx^i$$

$$\boxed{\int_\gamma df = f(b) - f(a)} \longrightarrow \int_a^b \nabla f \cdot d\vec{s} = f(b) - f(a)$$



3 Find a differential form on S^1 which is closed, but not exact.

closed form ω:

$$d\omega = 0$$

exact form ω_r:

$$\omega_r = d\alpha_{r-1}$$

By earlier proposition: ω exact $\Longrightarrow$ ω closed

$$d\omega_r = d(d\alpha_{r-1}) = 0$$

Is every closed form exact?

- ALMOST, up to "topology" [cohomology]

Example: manifold S^1: $\theta \sim \theta + 2\pi$. Consider $\omega = d\theta$ [S^1 is 1-d]

Trivially closed: $d\omega = 0$

Is it exact?

S^1

$$\int_\gamma \omega = \int_{\theta=0}^{\theta=2\pi} d\theta = 2\pi$$

If exact:

$$\int_0^{2\pi} d\alpha = \int_0^{2\pi} \frac{d\alpha}{d\theta} d\theta$$

0-form

$$= \alpha(2\pi) - \alpha(0) = 0$$

α single-valued

4 Combine the electric and magnetic fields into a 2-form F. What are electric and magnetic fluxes?

[$\Delta x^2 - \Delta t^2 =$ invariant]

$$F_{\mu\nu} = \begin{array}{c c} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} & \begin{array}{c} t \\ x \\ y \\ z \end{array} \\ \begin{array}{cccc} t & x & y & z \end{array} & \end{array}$$

$\Big\downarrow$ 2-form??

$$F = -E_x\, dt\wedge dx - E_y\, dt\wedge dy - E_z\, dt\wedge dz + B_x\, dy\wedge dz + B_y\, dz\wedge dx + B_z\, dx\wedge dy$$

magnetic flux through spatial surface S

$$\int_S F \longmapsto \int B_x\, dy\wedge dz + \cdots$$

$$\downarrow$$

$$\int B_x\, da_x + \cdots = \int_S \vec{B}\cdot d\vec{a}$$

$$-\int_S *F = \text{electric flux}$$

$$* \, dt\wedge dx = dy\wedge dz$$

$$[\, * \, dy\wedge dz = -dt\wedge dx \,]$$

$$* \quad E \rightleftarrows B$$

5 Write Maxwell's equations using differential forms.

$\nabla \cdot \vec{E} = \rho \qquad \boxed{\nabla \cdot \vec{B}} = 0 \qquad \nabla \times \vec{E} = -\partial_t \vec{B} \qquad \nabla \times \vec{B} = \vec{J} + \partial_t \vec{E}$

↳ apply exterior derivative! x comp $\nabla \times \vec{E} + \partial_t \vec{B} = 0$

$$dF = -d(E_x\, dt \wedge dx) - \cdots + d(B_x\, dy \wedge dz) + \cdots$$

$$= -\partial_y E_x\, dy \wedge dt \wedge dx - \underbrace{\cdots}_{5\text{ terms}} + \partial_t B_x\, dt \wedge dy \wedge dz + \partial_x B_x\, dx \wedge dy \wedge dz$$

$$+ \text{ 4 terms}$$

$$= dt \wedge dy \wedge dz \boxed{(\partial_t B_x - \partial_z E_y + \partial_y E_z)} + \text{ 2 terms}$$

$$+ dx \wedge dy \wedge dz \boxed{(\partial_x B_x + \partial_y B_y + \partial_z B_z)} \Rightarrow dF = 0$$

$$T^{\mu\nu} = F^{\mu}{}_{\rho} F^{\rho\nu} - \frac{1}{4} g^{\mu\nu} F^2$$

$$d{*}F = -dx \wedge dy \wedge dz\, (\partial_x E_x + \partial_y E_y + \partial_z E_z)$$

$$+ dt \wedge dy \wedge dz\, (-\partial_t E_x + \partial_z B_y - \partial_y B_z) + \text{ 2 terms} = -\rho\, dx \wedge dy \wedge dz$$

$$+ J_x\, dt \wedge dy \wedge dz + \cdots$$

$$d{*}F = {*}J$$

define $J = \rho\, dt + J_x dx + J_y dy + J_z dz$

6 State Stokes' Theorem in terms of differential forms.

Recall: in 3d ...

orientation

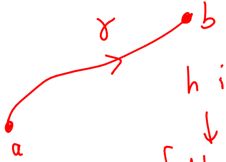
f is a 1-form

$$\int_S (\nabla \times \vec{f}) \cdot d\vec{a} = \oint_{\partial S} \vec{f} \cdot d\vec{s}$$

Using differential forms:

$$\int_S df = \int_{\partial S} f$$

This holds for any r-form f, and $(r+1)$-d surface S!



h is 0-form

Example: $\displaystyle\int_\gamma dh = h(b) - h(a)$

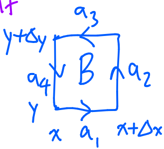
7 (Sketch) the proof of Stokes' Theorem.

[Sketch in 2d]

f is 1-form

df 2-form

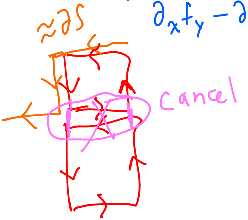
$$\int_S df = \sum_{\text{boxes } B} \int_B df$$



$$\int_B \underbrace{(df)_{xy}}_{\partial_x f_y - \partial_y f_x} \, dx \wedge dy = \int_x^{x+\Delta x} dx \int_y^{y+\Delta y} dy \, (\partial_x f_y - \partial_y f_x)$$

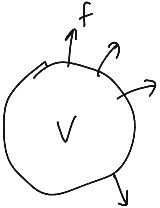
$$= \left[\int_y^{y+\Delta y} dy \, f_y \right]_x^{x+\Delta x} - \left[\int_x^{x+\Delta x} dx \, f_x \right]_y^{y+\Delta y}$$

$$= \int_{a_2} f + \int_{a_4} f + \int_{a_3} f + \int_{a_1} f = \int_{\partial B} f$$



$$\int_S df = \sum_{\text{boxes } B} \int_{\partial B} f = \int_{\partial S} f$$

8 Explain the divergence theorem using differential forms.



$$f = f_x\,dx + f_y\,dy + f_z\,dz$$

Divergence Thm: $$\int_V dV\, \nabla\cdot\vec{f} = \oint_{\partial V} d\vec{a}\cdot\vec{f}$$

3-d region

Using differential forms?

$$\int_V d{*f} = \int_{\partial V} {*f}$$

3d

2d

Stokes!