PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 2

January 19

1 Review the definition of groups, and group actions.

group G is a set together w/ "multiplication" operation:
(binary)

1) $g_1 g_2 \in G$

2) $(g_1 g_2) g_3 = g_1 (g_2 g_3) = g_1 g_2 g_3$

$\neq g_2 g_1 g_3$

in general

→ 3) $\exists$ 1 such that $1g = g1 = g$

"there exists" identity

→ 4) $\exists$ g^{-1} ; $gg^{-1} = g^{-1}g = 1$

such that

Proposition: $1 = 1^{-1}$.

Proof: 3) $1 \times g = g$

4) $1 \times 1^{-1} = 1$

$1^{-1} = 1^{-1} \times 1 = 1$

1×1^{-1}

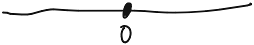
group G can act on a set X:

$(g_1 g_2) \overset{\text{acts on}}{\cdot} x = g_1 \cdot (g_2 \cdot x)$

G X

2 Describe how $\mathbb{R}$ is a group under addition. What about multiplication?

$\mathbb{R} = \{\text{real \#s } -\infty < x < \infty\}$



$\mathbb{R}$ is a group when operation is addition:

1) $x + y \in \mathbb{R}$

2) $x + y + z$ makes sense

3) $x + \underset{\text{identity}}{0} = x$

4) $x + \underset{\text{inverse}}{(-x)} = 0$

$x + y = y + x$ [Abelian group: $g_1 g_2 = g_2 g_1$]

Since $0 \times y = 0$ for any $y \in \mathbb{R}$ ... $(\underbrace{0^{-1} \times 0}_{1}) \times y = y$ doesn't work

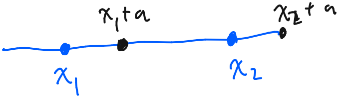
If we remove 0 ... $\mathbb{R} - \{0\} = \{x \in \mathbb{R} : x \neq 0\}$ is group under multiplication

$:$ such that

$\mathbb{R}^{\times}$

inverse $= \frac{1}{x}$ identity $= 1.000$

3 Write down the simplest effective Lagrangian for 2 interacting particles in one dimension, assuming translation invariance and a stable equilibrium. This is a toy model for a chemical bond.



translation invariant

symmetry group $\mathbb{R}$

translate by $a \in \mathbb{R}$: group action

$a \cdot x_1 = x_1 + a \qquad a \cdot x_2 = x_2 + a$

"~~physics~~" [not config of particles] is invariant

the Lagrangian [or even just action $\int dt\, L$) to be invariant

build L out of invariant objects

$L(x_1, x_2, \dot{x}_1, \dot{x}_2)$

$a\cdot(x_1 - x_2) = x_1 + a - (x_2+a) = x_1 - x_2$

$\dot{x}_1,\ \dot{x}_2,\ x_1 - x_2$ invariants

simplest possible L . . .

$$\boxed{L = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 - \frac{k}{2}(x_1 - x_2)^2 \ \ + a\dot{x}_1}$$

doesn't

$L \to L + \frac{d}{dt} f$ change EOM

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1}$

$+\frac{d}{dt}a$

4 Describe the groups $\mathbb{Z}$ and $\mathbb{Z}_n$.

$\mathbb{Z}$ = integers = $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\frac{3}{2} \notin \mathbb{Z}$

- group under addition: $3 + 0 = 3$
  $3 + (-3) = 0$

$\mathbb{Z}_n = \{\overset{\text{id}}{0}, 1, 2, \ldots, n-1\}$ (n elements)

is a group under addition (mod n)

E.g. $n = 5$: ($\mathbb{Z}_5$)

$4 + 0 = 0 + 4 = 4 \pmod 5$

$4 + 4 = 8$
$= 5 + 3$
$= 3 \pmod 5$

$a = b + nk \pmod n$

$k \in \mathbb{Z}$

5 Define and explain an equivalence class.

equivalence relation ($\sim$): a way to compare 2 elements of a set X

1) $a \sim a$
2) $a \sim b$ then $b \sim a$
3) $a \sim b$ and $b \sim c \Rightarrow a \sim c$

equivalence class:

$[a] = \{a' \in X : a' \sim a\}$

modulo: $X/_\sim = \{[a], [b], \ldots\}$

$\mathbb{Z}/\sim = \{[0], [1], [2], [3], [4]\} = \mathbb{Z}_5$

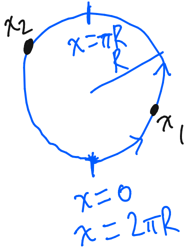
E.g. $X = \mathbb{Z}$ and $a \sim b$ if $a - b = 5k$, $k \in \mathbb{Z}$

$0 \sim 5 \sim 10 \ldots$

$\ldots, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$

$[0] = \{0, 5, 10, \ldots, -5, -10, \ldots\}$

6 Write down the simplest Lagrangian for 2 interacting particles on a circle, assuming an appropriate translation invariance. What changes when the space is a circle?



define equivalence relation

$x_1 \sim x_2$ if $x_1 - x_2 = 2\pi R n$, $n \in \mathbb{Z}$

circle $= S^1 = \mathbb{R}/\sim \; = \mathbb{R}/\mathbb{Z}$

translation symmetry: $a \cdot x_1 = x_1 + a$
$a \cdot x_2 = x_2 + a$

but $a \in \mathbb{R}/\mathbb{Z}$ ← also a group under addition

$0.3 + 2.5 = 2.8$
$\tilde{0.3} + \tilde{0.5} = \tilde{0.8}$

most general Lagrangian?

$$L(\dot{x}_1, \dot{x}_2, x_1 - x_2)$$
$$= L(\dot{x}_1, \dot{x}_2, x_1 - x_2 + 2\pi R)$$

e.g.

$$L = \frac{m}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + U\cos\frac{x_1 - x_2}{R}$$

7 Define the groups O(n) and SO(n). orthogonal groups

$$O(n) = \{ V \in \mathbb{R}^{n \times n} : V^T V = 1 \}$$

$\mathbb{R}^{n\times n}$: n×n matrix w/ real coeffs

$:$ such that

1: n×n identity matrix

check it's group?

1) $(V_1V_2)^T(V_1V_2) = V_2^T V_1^T V_1 V_2 = V_2^T \cancel{1} V_2 = 1$ ✓

3) $1^T 1 = 1 \quad (1^T = 1)$

4) $V^{-1} = V^T \quad ([V^TV=1]^T \Rightarrow VV^T = 1$

<u>Proposition:</u> $V \in O(n) \Rightarrow \det(V) = \pm 1$

Proof: $\det(V^TV) = \det(V)^2 = \det(1) = 1$

$$SO(n) = \{ V \in O(n) : \det(V) = 1 \}$$

SO: special orthogonal

8 Write down the simplest Lagrangian for two interacting particles, assuming translation and rotation invariance.

d spatial dimensions

rotation symmetry group $SO(3)$

rotation preserves length:

$$\vec{x} \cdot \vec{x} = V\vec{x} \cdot V\vec{x}$$

"rotation matrix"

$$\vec{x}^T \vec{x} = \vec{x}^T \underbrace{V^T V}_{1} \vec{x}$$

$SO(3)$ not $O(3)$ b/c...:

$$V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad [\det = -1]$$

not a rotation

9 Define the groups U(n) and SU(n). unitary groups

$$U(n) = \{ V \in \mathbb{C}^{n\times n} \; : \; V^\dagger V = 1 \}$$

complex

Hermitian conjugate

$$V^\dagger = (V^T)^* \quad \text{complex conj-}$$

$$= \overline{(V^T)}$$

$$SU(n) = \{ V \in U(n) \; : \; \det(V) = 1 \}.$$