PHYS 5040
Algebra and Topology in Physics
Spring 2021

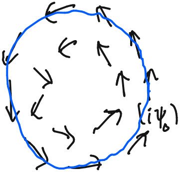
Lecture 20

March 23

1 Give physical examples of "topological defects".

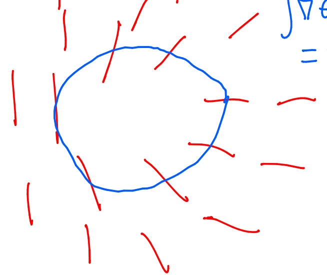
superfluid vortex:

[recall: $\psi = \psi_0 e^{i\theta}$]



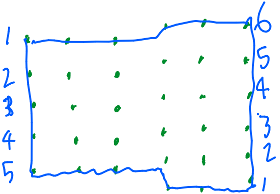
$\int \nabla\theta \cdot d\vec{s} = 2\pi n$, $n \in \mathbb{Z}$

liquid crystals: "$\int \nabla\theta \cdot d\vec{s}$" $= \pi$

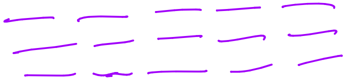


(solid-state) crystals:

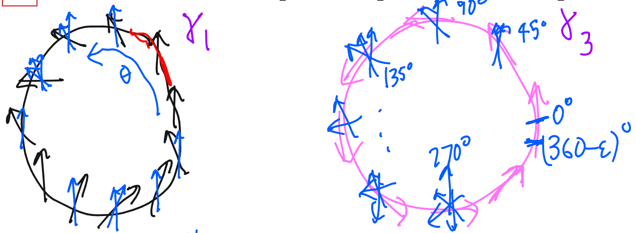
$6-5=1$



e.g. LC equilibrium:



2 When are two loops in a space X homotopic?



not topologically protected

topological

$\gamma: S^1 \mapsto X$ (order parameter)

[e.g. SF, $X = S^1$]

Loops γ_1 & γ_2 are homotopic [or homotopy equivalent] if $\exists$ continuous function $\Gamma: S^1 \times [0,1] \to X$ s.t. $\Gamma(\theta, 0) = \gamma_1(\theta)$, $\Gamma(\theta, 1) = \gamma_2(\theta)$ (with $t \in [0,1]$)

Example:

3 Define the group $\pi_1(X, x_0)$.

Equivalence relation on maps $\gamma: S^1 \to X$:

$\gamma_1 \sim \gamma_2$ if γ_1 & γ_2 are homotopic

1) $\gamma_1 \sim \gamma_1$

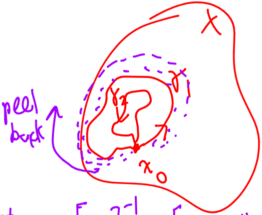
2) $\gamma_1 \sim \gamma_2 \Leftrightarrow \gamma_2 \sim \gamma_1$ $\quad \Gamma(t) \to \Gamma(1-t)$

3) $\gamma_1 \sim \gamma_2$ & $\gamma_2 \sim \gamma_3 \Rightarrow \gamma_1 \sim \gamma_3$: $\Gamma_{1\to 3}(t) = \begin{cases} \Gamma_{1\to 2}(2t) & t < \frac{1}{2} \\ \Gamma_{2\to 3}(2t-1) & t > \frac{1}{2} \end{cases}$

equivalence classes: $[\gamma] = \{\text{all } \gamma_j : \gamma_j \sim \gamma\}$

Natural "multiplication" on these $[\gamma]$:

$[\gamma_1][\gamma_2] = [\gamma_2 \text{ then } \gamma_1]$



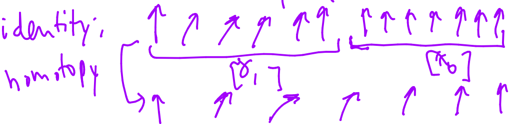
let's assume $\gamma_{1,2}$ start and end at $x_0 \in X$

sit at x_0

$[\gamma_1]^{-1} = [\gamma_1 \text{ run backwards}]$

Without equivalence class... $\gamma_1 \gamma_* \stackrel{?}{=} x_0$

$[x_0]$ is the identity:

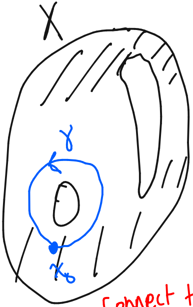


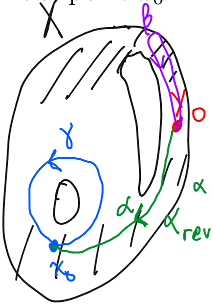
$\begin{cases} \gamma_1(2\theta) \\ x_0 \end{cases}$

$\begin{cases} \gamma_1((2-t)\theta) \\ x_0 \end{cases}$

4 Show that if the space X is connected, then $\pi_1(X, x_0) = \pi_1(X)$ does not depend on the initial point x_0.

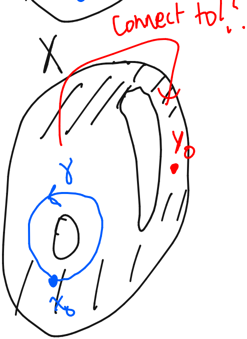
$\pi_1(X)$: fundamental group





α from $y_0 \to x_0$

α_{rev} goes $x_0 \to y_0$



Find ρ^{-1}:

$\rho^{-1}([\beta]) = [\alpha \beta \alpha_{rev}]$

$\rho(\rho^{-1}([\beta])) \doteq [\beta]$

homomorphism ρ from $\pi_1(X, x_0) \to \pi_1(X, y_0)$

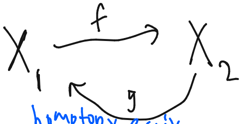
ρ invertible $\Rightarrow \pi_1(X, x_0) \cong \pi_1(X, y_0)$ (isomorphic)

$\rho([\gamma]) = [\alpha_{rev} \gamma \alpha]$ (α_{rev} then γ then α)

$\rho([\gamma_1][\gamma_2]) = [\alpha_{rev} \gamma_1 \gamma_2 \alpha]$

$= [\alpha_{rev} \gamma_1 \alpha \alpha_{rev} \gamma_2 \gamma_1] = \rho([\gamma_1]) \cdot \rho([\gamma_2])$

5 When are two spaces homotopy equivalent? What is a deformation retraction?



homotopy equiv

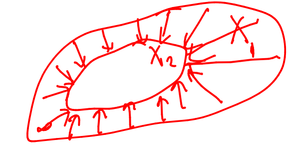
identity

$X_1 \sim X_2$ if $f \circ g \sim id_{X_2}$ and $g \circ f \sim id_{X_1}$

???

two functions $F_1, F_2 : X_3 \to X_4$ are homotopic if $\exists$ continuous $\tilde{F} : X_3 \times [0,1] \to X_4$ such that $\tilde{F}(x,0) = F_3(x)$, $\tilde{F}(x,1) = F_4(x)$

Example: deformation retraction



$f : X_1 \to X_2$ pulls points not in X_2 to closest point in X_2

$g : X_2 \to X_1$ takes $x \to x$

$f \circ g = id_{X_2}$ $\qquad$ $g \circ f$ "=" $f \sim id_{X_1}$

6 Can $\mathbb{R}^n$ be contracted to a point? What about $\mathbb{R}^n - \{0\}$?

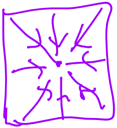
$\mathbb{R}^n = \{(x_1, \ldots, x_n)\}$ ($\in \mathbb{R}$)

$\tilde{F} : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ (t)

such that $\tilde{F}(\vec{x}, 1) = \vec{0}$

$\tilde{F}(\vec{x}, 0) = \vec{x}$??

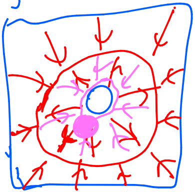
$\tilde{F}(\vec{x}, t) = (1-t)\vec{x}$



$\mathbb{R}^n$ is homotopy equivalent to a single point

→ contractible space

$\mathbb{R}^n - \{0\}$ [delete origin]



[def.] retracted onto S^{n-1}

how to prove this?

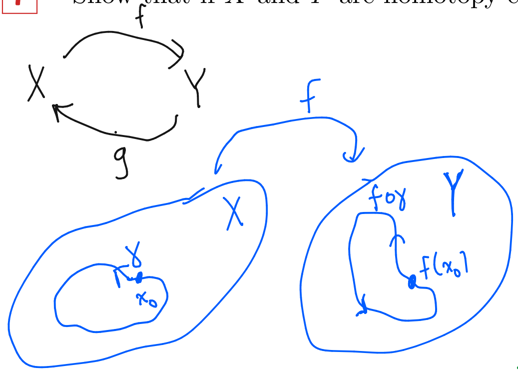
Step 1: deform. retract to simplest possible space $[S^{n-1}]$

Step 2: e.g. if $n=2$,

$\pi_1(S^1) = \mathbb{Z}$

$[\pi_n(S^n) = \mathbb{Z}]$

7 Show that if X and Y are homotopy equivalent, then $\pi_1(X) = \pi_1(Y)$.



$f_*: \pi_1(X) \to \pi_1(Y)$

↑ homomorphism!

loops in Y

$f_*([\gamma]) = [f \circ \gamma]$

$g_*: \pi_1(Y) \to \pi_1(X)$

$g_*(f_*([\gamma]))$

$= g_*([f \circ \gamma])$

$= [g \circ f \circ \gamma]$

$= [\gamma]$

$g \circ f \sim id_X$ if f & g are homotopic

$g_* = f_*^{-1}$, conclude $\pi_1(X) = \pi_1(Y)$