PHYS 5040

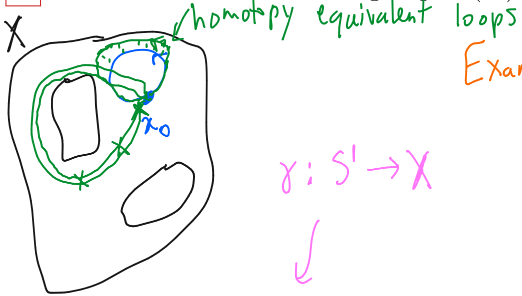
Algebra and Topology in Physics

Spring 2021

Lecture 21

March 30

1 Review the fundamental group $\pi_1(X)$.



$\gamma : S^1 \to X$

equivalence relation:

$\gamma_1 \sim \gamma_2$ if homotopy
(loop)

equivalence classes: $\in \pi_1(X)$

$[\gamma_1] = \{\gamma \text{ loop} : \gamma \sim \gamma_1\}$

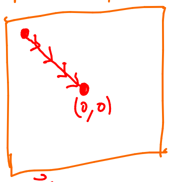
$[\gamma_1][\gamma_2] = [\gamma_2 \text{ then } \gamma_1]$

Example: $\pi_1(\mathbb{R}^n) = \pi_1(\text{point})$

$= \{\text{identity}\}$

(trivial group)

$\pi_1(\text{point}) = 0$



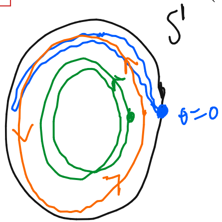
$\vec{f}(\vec{x}, t) = (1-t)\vec{x}$
$\mathbb{R}^2$, $[0,1]$

retracts $\mathbb{R}^2$ to $(0,0)$
(homotopic)

$[\gamma]^{-1} = [\gamma \text{ traversed backwards}]$

identity $= [\text{loop sits at } x_0]$

2 Prove that $\pi_1(S^1) = \mathbb{Z}$.



parameterize by $\phi \sim \phi + 2\pi$

$\gamma(\theta) = \theta$

$\Downarrow$

$n = 1$

loop w/ winding # = 2
$\gamma(\theta) = 2\theta$

$\gamma : \underset{\theta}{S^1} \to \underset{\gamma(\theta),\ \phi}{S^1}$

$\gamma(\theta) + 2\pi \sim \gamma(\theta) = \gamma(\theta + 2\pi)$

Define winding number:

$$n(\gamma) := \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \frac{d\gamma}{d\theta}$$

1) Suppose we relax $\gamma(\theta) = \gamma(\theta + 2\pi)$
AND $\gamma(\theta) \sim \gamma(\theta) + 2\pi$

then $\gamma : \underset{\theta}{\mathbb{R}} \to \underset{\gamma(\theta)}{\mathbb{R}}$

Then $n(\gamma) = \frac{1}{2\pi} \int_{\gamma(0)}^{\gamma(2\pi)} d\gamma = \frac{\gamma(2\pi) - \gamma(0)}{2\pi}$

2) Now impose $\sim$:

$$\frac{\gamma(2\pi) - \gamma(0)}{2\pi} = \frac{2\pi \cdot \tilde{n}}{2\pi} = \tilde{n}, \quad \tilde{n} \in \mathbb{Z}$$

Emphasize: $\gamma(2\pi) \sim \gamma(0)$

3) If $\gamma_1 \overset{\text{homotopic}}{\sim} \gamma_2$, $n(\gamma_1) = n(\gamma_2)$.

$\Rightarrow$ existence smooth $\Gamma(\theta, t)$
such that $\Gamma(\theta, 0) = \gamma_1(\theta)$, $\Gamma(\theta, 1) = \gamma_2$

- only continuous map [in t] into discrete space [$\mathbb{Z}$] is constant

4) $n(\gamma_1) + n(\gamma_2) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{d\gamma_1(\theta)}{d\theta} + \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{d\gamma_2(\theta)}{d\theta}$

$$= \frac{1}{2\pi} \left[\int_0^{\pi} d\tilde{\theta} \frac{d\gamma_1(2\tilde{\theta})}{d\tilde{\theta}} + \int_{\pi}^{2\pi} d\tilde{\theta} \frac{d\gamma_2(2\tilde{\theta} - \pi)}{d\tilde{\theta}} \right]$$

$$= n([\gamma_2][\gamma_1]) \quad \checkmark$$

Check: if $n(\gamma) = 0 \ldots$

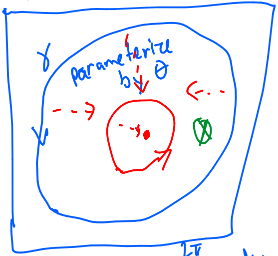
$\Gamma(\theta, t) = (1 - t)\gamma(\theta)$ homotopy to a point

3 Show that superfluid vortices must be quantized. What do they look like in three dimensions?

Recall: in superfluid, low energy configs characterized by a map

$$\phi : \{\text{spatial domain}\} \to S^1$$

two dimensions:



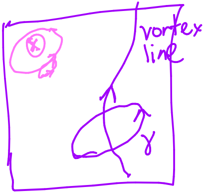
topological defect: vortex

Evaluate $n = \frac{1}{2\pi}\int_0^{2\pi} d\theta \, \frac{d\gamma}{d\theta} \neq 0$

$\in \mathbb{Z}$

If superfluid exists everywhere...
n would be the same for γ as for loop traversing a point, w/ zero winding #

three dimensions...



$$\pi_1(\mathbb{R}^3 - \text{line}) = \pi_1(\mathbb{R}^2 - \text{point})$$
$$= \pi_1(S^1) = \mathbb{Z}$$

in general, if you have theory in d dimensions, if $\pi(S^1) \neq 0$, topological defects which are $(d-2)$-dim surfaces

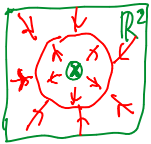
4 State (and prove) the Borsuk-Ulam Theorem (in two dimensions).

If $\vec{f}: S^2 \to \mathbb{R}^2$, $\exists\ x \in S^2$ where $\vec{f}(x) = \vec{f}(-x)$ antipodal

[$\exists$ a point on Earth where T, P are same as on opposite point on Earth]

Proof: Suppose statement is false. Then $\vec{f}(x) - \vec{f}(-\vec{x}) \neq \vec{0}$

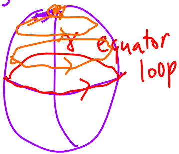
define on $\mathbb{R}^2 - (0,0)$



$g: S^2 \to S^1$

$$g(x) = \frac{\vec{f}(x) - \vec{f}(-x)}{|\vec{f}(x) - \vec{f}(-x)|} = (\cos\phi, \sin\phi)$$

S^2



$g|_\gamma = g_\gamma : S^1 \to S^1$

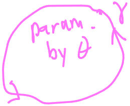
Winding #1: contract γ to north pole

$[\pi_1(S^2) = 0]$

$n(g_\gamma) = 0$

denote $g_\gamma = \phi$

Winding #2:



$g_\gamma(\theta) \sim g_\gamma(\theta + \pi) + \pi$

extend to $\mathbb{R}$

$g_\gamma(\theta) = g_\gamma(\theta + \pi) + \tilde{n}\pi$ odd integer

$g_\gamma(\theta + 2\pi) = g_\gamma(\theta) + 2\pi\tilde{n}$

$n(g_\gamma) = \tilde{n}$

Contradiction: $\tilde{n} = 0$.

5 Show that $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

$X \times Y$ is the set of (x, y)

$x \in X$, $y \in Y$

Proof: define $p_X : X \times Y \to X$, similar $p_Y : X \times Y \to Y$

$p_X(x,y) = x$

induce

homomorphism $\rho : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$

$\rho([\gamma]) = ([p_X \circ \gamma], [p_Y \circ \gamma])$

$\gamma = (\gamma_X(\theta), \gamma_Y(\theta))$

then

$p_X \circ \gamma = \gamma_X$

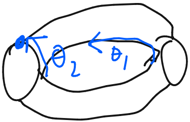
find $\varphi : \pi_1(X) \times \pi_1(Y) \to \pi_1(X \times Y)$

$\varphi(([\gamma_X], [\gamma_Y])) = [(\gamma_X, \gamma_Y)]$

$\rho = \varphi^{-1}$.

6 Find the fundamental group of the torus.

Recall: 2-dimensional torus (T^2)



$$T^2 = \underset{\theta_1}{S^1} \times \underset{\theta_2}{S^1}$$

$$\pi_1(T^2) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$$

but $\pi_1(S^2) = 0$