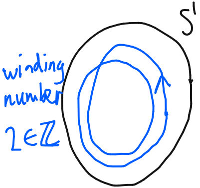
PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 22

April 1

1 What is a universal covering space?

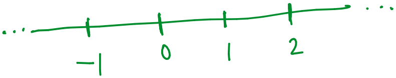
Recall: $\pi_1(S^1) = \mathbb{Z}$

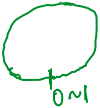


Recall: $S^1 = \mathbb{R}/\mathbb{Z}$

$n \in \mathbb{Z}$: $x \longmapsto x+n$

under modulo: $x \sim x+1$





Today: • $S^1 = \mathbb{R}/\mathbb{Z}$

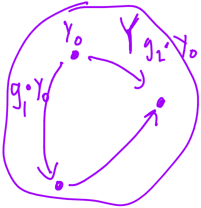
• $\pi_1(\mathbb{R}) = 0$ ← simply connected

• free action: $x \sim x+1$ has no "fixed points" where $x = x+1$ already

$\Rightarrow \mathbb{R}$ is a universal cover(ing space) for S^1, and $\pi_1(S^1) = \mathbb{Z}$

Take Y simply connected: $\pi_1(Y) = 0$

group G acts on Y

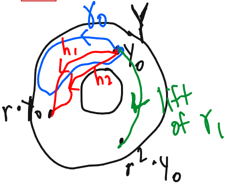


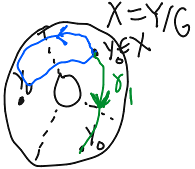
If $g_1 \cdot y_0 = g_2 \cdot y_0 \iff g_1 = g_2 \quad \forall y_0$

then G acts freely on Y

Quotient space $X = Y/G \qquad \{y_0, g_1 \cdot y_0, \dots\} \in X$

2 Explain why for simply connected space Y, $\pi_1(\underbrace{Y/G}_{X}) = G$.





If γ_0 is loop in Y, then it corresponds to trivial loop in X [homotopy exists in Y ⇒ exists in X, continuous...]

$\pi_1(X) \leq G$:

if not, ∃ $h_1, h_2 \in \pi_1(X)$ both end at same point in Y.

Then $h_2^{-1} h_1$ is contractible in Y ⇒ induces reverse homotopy to point in X, $h_2^{-1} h_1 = \text{identity}$, so $h_1 = h_2$

If γ_1 became loop in Y, then homotopy equiv to point (in both X & Y), But lift of γ_1 connects disjoint points, regardless of start/end (G acts freely)

If γ_1 were homotopic to point, then $\Gamma(\theta, t)$ homotopy $\Gamma(\theta, 0) = \gamma_1$ ⇒ lift in Y into homotopy... but for some t_1 endpoint in Y has to jump (not allowed). $\Gamma(\theta, 1) = y_0$

This holds ∀ pair of points $g \cdot y_0$, so we conclude $G \leq \pi_1(X)$

$y_0 \curvearrowright g_1 \cdot y_0 \curvearrowright g_2 \cdot (g_1 \cdot y_0)$

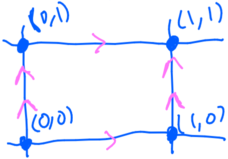
$g_1 \in \pi_1$ $g_2 \in \pi_1$

3 Describe the classification of point topological defects in crystalline solids in two dimensions.

$$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$$

$$T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \qquad (n_1, n_2) \in \mathbb{Z}^2$$

$$\underbrace{(x_1, x_2)}_{\in \mathbb{R}^2} \sim (x_1 + n_1, x_2 + n_2)$$



• additive group action is free

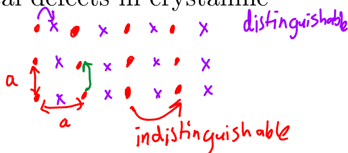
$$\vec{n} \cdot \vec{x} = \vec{x} + \vec{n} \neq \vec{x} \text{ if } \vec{n} \neq \vec{0}$$

group action

• $\pi_1(\mathbb{R}^2) = 0$

$\Rightarrow \pi_1(T^2) = \mathbb{Z}^2$

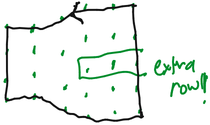
Crystal: (square lattice)



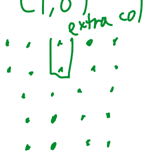
Classify equilibria of crystal by T^2 [in two dimension]

topological defects:

(0,1)



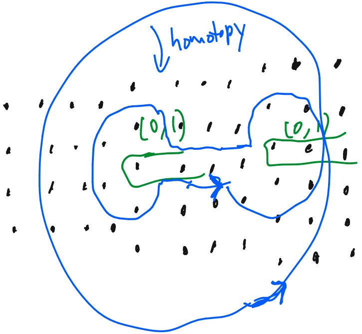
(1,0)

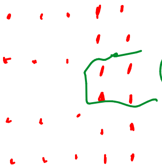


4 Show how to "add" 2 topological defects in a solid – what does that correspond to in real space?

global defect

$(0,2) = (0,1) + (0,1)$





(0,2) defect microscopically???

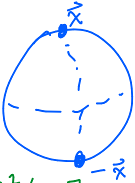
this will not [generally] arise physically: energetically unfavorable

5 Calculate $\pi_1(\mathbb{RP}^2)$.

Recall: $\mathbb{RP}^2 = \mathbb{R}^3 - \{0\} / [\vec{x} \sim \lambda \vec{x},\ \lambda \neq 0]$ (set of all lines passing thru origin)

nematic liquid crystal

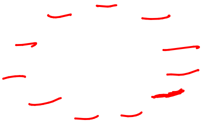
On HW 8, $\mathbb{RP}^2 = S^2 / [\vec{x} \sim -\vec{x}]$



- group $G = \{\pm 1\} = \mathbb{Z}_2$, acts <u>freely</u> $[\mathbb{RP}^2 = S^2/\mathbb{Z}_2]$
- $\pi_1(S^2) = 0$

$\Rightarrow \pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$

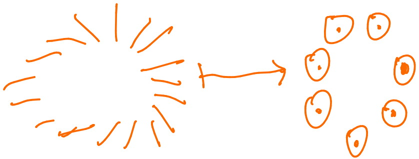
topological "defects":



$\{1\}$

2 "-1" defects:

$(-1)\times(-1) = 1$



$\{-1\}$

6 Calculate $\pi_1(\mathrm{SO}(3))$.

$SO(3) = \{ M \in \mathbb{R}^{3\times 3} : M^T M = \mathbb{1},\ \det(M) = 1 \}$

$SO(3) = SU(2)/\mathbb{Z}_2$

$SU(2) = \{ M \in \mathbb{C}^{2\times 2} : M^\dagger M = \mathbb{1},\ \det(M) = 1 \}$

$\mathbb{Z}^2 = \{ \pm \mathbb{1} \}$ identity matrix

SU(2) is universal cover of SO(3).

Why is $\pi_1(SU(2)) = 0$?

HW1: $SU(2) = \{ a\mathbb{1} + b \cdot i\sigma^x + c \cdot i\sigma^y + d \cdot i\sigma^z : \underbrace{a^2 + b^2 + c^2 + d^2 = 1}_{S^3 \subset \mathbb{R}^4} \}$

$SU(2) = S^3$

$\pi_1(S^3) = 0$ [same reason as S^2]

$\Rightarrow \pi_1(SO(3)) = \mathbb{Z}_2$