PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 23

April 6

1 Define a free group, and a free product of groups.

Free product: groups G and H

$1, g_1, \dots$ $\qquad$ $1, h_1, h_2, \dots$

$G*H = \{1, g_1, \dots, h_1, \dots, g_1h_1, \dots, h_1g_1, \dots, g_1hg, \dots, hgh, \dots\}$

multiplication: $\qquad (h_3 = h_1h_2)$

$$(g_1h_1)h_2 = g_1h_3$$
$$(g_1h_1)g_2 = g_1h_1g_2$$

inverse: $(g_1h_1)^{-1} = h_1^{-1}g_1^{-1}$: $(g_1h_1)(h_1^{-1}g_1^{-1}) = g_1g_1^{-1} = 1$

free group: free product of $\mathbb{Z}$: $\mathbb{Z}, \mathbb{Z}*\mathbb{Z}, \mathbb{Z}*\mathbb{Z}*\mathbb{Z}, \dots$

quotient a free group... $\underset{(r)}{\mathbb{Z}} * \underset{(s)}{\mathbb{Z}} = \langle r, s\rangle$

$$D_6 = \langle r, s \mid r^3 = s^2 = rsr^{-2}s^{-1} = 1\rangle$$
$$= \mathbb{Z}*\mathbb{Z} / \langle r^3, s^2, rsr^{-2}s^{-1}\rangle$$

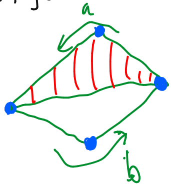
$\underset{(a)}{\mathbb{Z}} * \underset{(b)}{\mathbb{Z}} * \underset{(c)}{\mathbb{Z}}$

$a, a^{-1} \quad b, b^{-1} \quad c, c^{-1}$

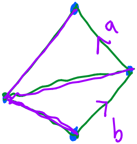
$abbbcac^{-1}c^{-1}\cancel{aa^{-1}}bb\dots$

2 Explain how to compute $\pi_1(X)$ for any CW complex X.

Algorithm for $\pi_1(X)$, if X is a CW-complex: van Kampen's Thm:

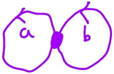


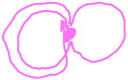
0-cells
1-cells
2-cell



homotopic to

X^1





a b a

$\pi_1(X^1) = \underset{(a)}{\mathbb{Z}} * \underset{(b)}{\mathbb{Z}}$

Add 2-cells:

Now, loop a is contractible

$\pi_1(X) = \pi_1(X^1) / \langle 1, a, a^2, \dots \rangle$

Take $\underset{(a)}{\mathbb{Z}} * \underset{(b)}{\mathbb{Z}}$: for each element, replace a, a^{-1} w/ 1

$\pi_1(X) = \underset{(b)}{\mathbb{Z}}$

van Kampen's: given CW complex X,

$\pi_1(X) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}} / N$

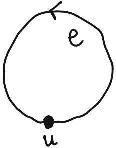
one copy for each loop in X^1 (1-skeleton)

N = generated by words corresp. to ∂(2-cell)

3 Calculate $\pi_1(S^n)$. $[n \geq 1]$

Case: $n=1$.

S^1:

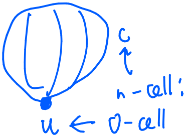


1-skeleton has 1 loop:

$\pi_1(S^1) = \mathbb{Z}$

S^1 is = 1-dim skeleton

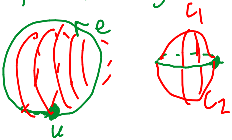
Case $n>1$:



attach c via $\partial c \sim u$

no 1-skeleton: $\pi_1(S^n) = 0$

Re-do: S^2

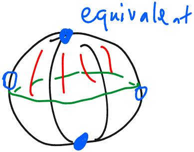


$\partial c_1 = e$
$\partial c_2 = e$

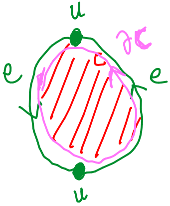
$\pi_1(S^2) = \langle e \rangle / \langle e \rangle = 0$

$\mathbb{Z}$
(e)

4 Calculate the fundamental group of $\mathbb{RP}^2$.



Recall: (HW 8) $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$



Add 2-cell c: $\partial c = e^2$

Loops in 1-skeleton: e (1 loop)

$$\pi_1(\mathbb{RP}^2) = \langle e\rangle / \langle e^2\rangle = \mathbb{Z}_2$$

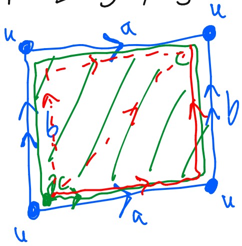
$$= \{1, e\}$$

$e^2 \to 1$

$e^3 = e\cancel{e}$

5 Calculate the fundamental group of two-dimensional torus, and the Klein bottle.

$T^2 = S^1 \times S^1$



$\partial c = a b a^{-1} b^{-1}$

$\pi_1(\text{1-skeleton}) = \underset{(a)}{\mathbb{Z}} * \underset{(b)}{\mathbb{Z}} = \langle a, b \rangle$

$\pi_1(T^2) = \langle a, b \rangle / \langle a b a^{-1} b^{-1} \rangle = \mathbb{Z} \times \mathbb{Z}$

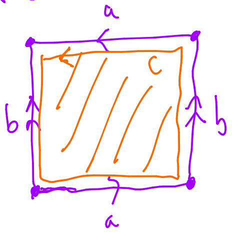
$a b a^{-1} b^{-1} = 1$

$a b a^{-1} = b$

$a b = b a$

$a^n b^m$

Klein bottle:



$\partial c = a b a b^{-1}$

$\pi_1(K) = \langle a, b \rangle / \langle a b a b^{-1} \rangle$

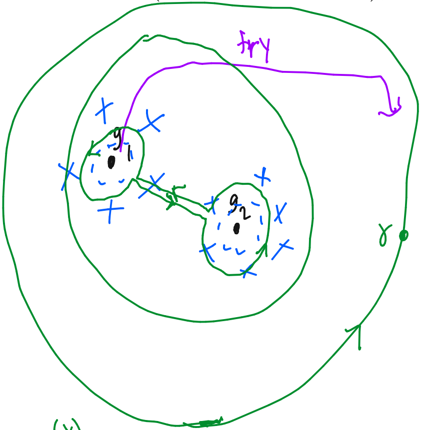
$a b a = b$

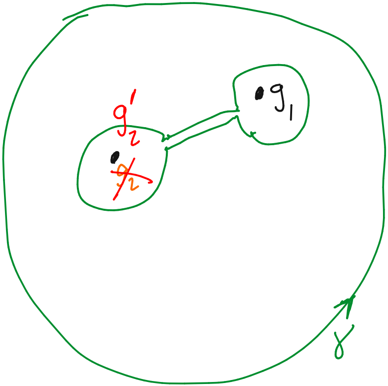
$\pi_1(K)$ is not Abelian group:

$a b = b a^{-1} \quad (a \neq a^{-1})$

Since $\pi_1(K) \neq \pi_1(T^2)$, these spaces are not homeomorphic.

6 What happens when you move two topological defects around each other (in two dimensions), when $\pi_1(X)$ is non-Abelian?





$\pi_1(X) \ni [\gamma] = g_1 g_2$

$[\gamma] = \cancel{g_2 g_1} \quad g_2' g_1$

$\Rightarrow g_1 g_2 = g_2 g_1$ [not if $\pi_1(X)$ not Abel.

7

Preview $\pi_n(X)$.