PHYS 5040
Algebra and Topology in Physics
Spring 2021

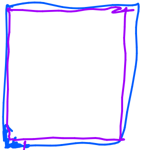
Lecture 24

April 8

1 Define $\pi_n(X)$.

$\pi_n(X)$ = group whose elements are $\gamma : S^n \to X$, up to homotopy equiv.

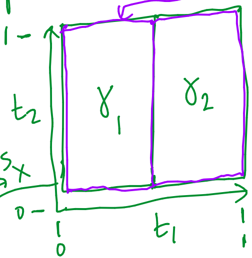
How to visualize: S^n is topologically equivalent to



↳ identify ∂ as single point

Multiplication in $\pi_n(X)$: γ_1 goes to $x_0 \in X$

$[\gamma_2][\gamma_1]$

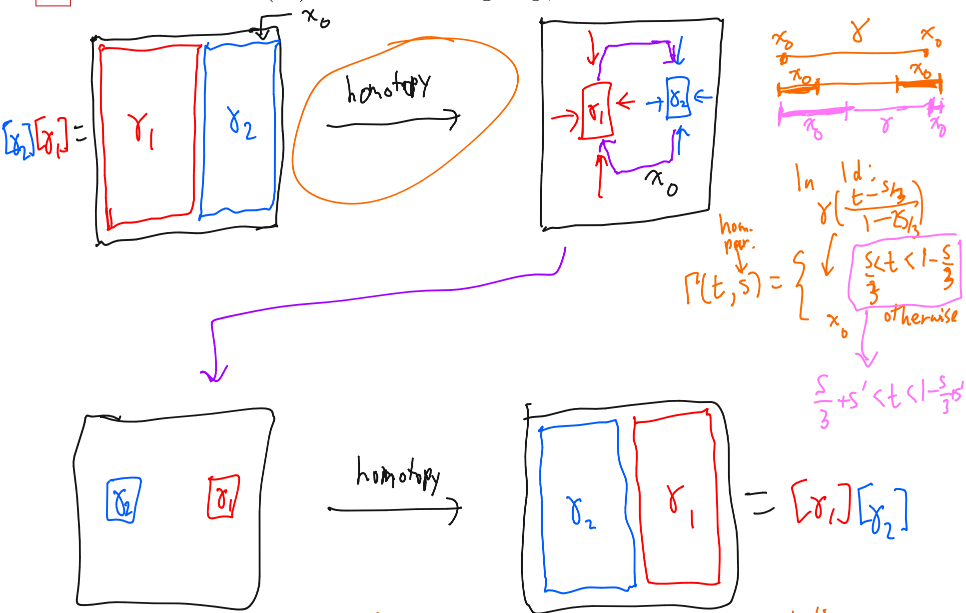


call this $\gamma_3 : S^n \to X$

$$\gamma_3(t_1, t_2) = \begin{cases} \gamma_1(2t_1, t_2) & t_1 < 1/2 \\ \gamma_2(2t_1 - 1, t_2) & t_1 > 1/2 \end{cases}$$

$\gamma_3(t_1, t_2) = x_0$ if $\underbrace{t_1 \text{ or } t_2 = 0 \text{ or } 1}_{\text{boundary}}$

2 Show that $\pi_n(X)$ is an Abelian group, if $n > 1$. [Illustrate for $n=2$]



Therefore, $\pi_n(X)$ is Abelian: multiplication is commutative

3 Discuss a number of other useful properties of $\pi_n(X)$:

1) $\pi_n(X)$ does not depend on x_0 (basepoint) if path connected

2) $\pi_n(X) = \pi_n(Y)$ if X & Y are homotopy equivalent

3) $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$

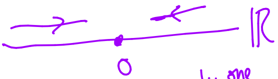
all proved similarly to what we did for π_1

$\pi_1(\tilde{X}) = 0$

4) if $\tilde{X}$ is a universal cover $[X = \tilde{X}/G]$, then:

- $\pi_1(X) = G$
- $\pi_n(X) = \pi_n(\tilde{X})$

$\Gamma(t,x) = tx$



e.g. $S^1 = \mathbb{R}/\mathbb{Z}$: $\pi_1(S^1) = \mathbb{Z}$

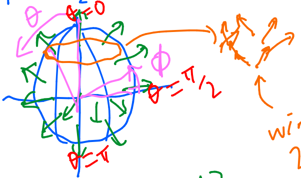
$\pi_n(S^1) \underset{\#4}{=} \pi_n(\mathbb{R}) \overset{\#2}{=} \pi_n(\text{point}) \overset{\text{only one map } S^n \to \text{pt.}}{=} 0$

4 Argue that $\pi_n(S^n) = \mathbb{Z}$.

Depict for $n=2$:

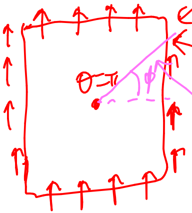
$1 \in \pi_2(X)$ $\qquad \gamma : S^2 \to S^2$

unit vectors in $\mathbb{R}^3$ $[x^2+y^2+z^2=1$ is $S^2 \subset \mathbb{R}^3]$



[radially outward]

wind in ϕ 2x on the $[\gamma]=2$ map



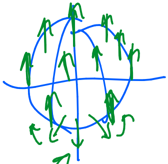
S^2 as square w/ ∂ identified

∂ as $\theta=0$

uniformly map points on this segment to $0<\theta<\pi$.

map $\gamma : S^2 \to S^2$ with $[\gamma] = n \in \pi_2(S^2) = \mathbb{Z}$

$$\gamma(\theta,\phi) = (\theta, n\phi)$$

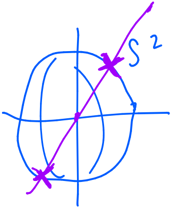


can't rotate S Pole to point up... picking ϕ direction to wind up $\Rightarrow$ discontinuity

5 Calculate $\pi_2(T^2)$, $\pi_2(\mathbb{R}P^2)$, and $\pi_2(S^2 \times S^2)$.

$$\pi_2(S^2 \times S^2) \underset{\#3}{=} \pi_2(S^2) \times \pi_2(S^2) = \mathbb{Z} \times \mathbb{Z}$$

$$\pi_2(\mathbb{R}P^2) = \pi_2(S^2/\mathbb{Z}_2) \underset{\#4}{=} \pi_2(S^2) = \mathbb{Z}$$



$$\pi_2(T^2) = \pi_2(S^1 \times S^1) \underset{\#3}{=} \pi_2(S^1) \times \pi_2(S^1) = 0$$

$$\pi_2(S^1) = \pi_2(\mathbb{R}/\mathbb{Z}) \underset{\#4}{=} 0$$

#4 b/c $\mathbb{R}$ is contractible

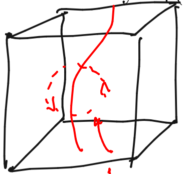
homotopy equiv to pt.

6 Describe what kinds of defects can be classified by $\pi_1(X)$, $\pi_2(X)$ and $\pi_3(X)$ in three dimensions.

$\pi_1(X)$: line defect

$3 - 1 - 1 = 1$

radial

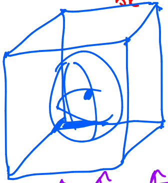


$\pi_1(\mathbb{R}^3 - \text{line}) = \pi_1(\mathbb{R}^2 - \text{point}) = \pi_1(S^1) \neq 0$

loop cannot be contracted

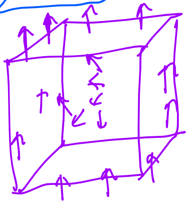
$\pi_2(X)$: point defect

$3 - 2 - 1 = 0$



$\mathbb{R}^3 - \text{point}$ homotopic to S^2

$\pi_3(X)$: smooth maps from $\mathbb{R}^3 \to X$; maps will tend to $x_0 \in X$ as $z \in \mathbb{R}^3$ goes to ∞ ...



7 A nematic liquid crystal has order parameter space $\mathbb{RP}^2$. Using that $\pi_3(\mathbb{RP}^2) = \mathbb{Z}$, propose a non-trivial topological defect in 3 dimensions corresponding to π_1, π_2 and π_3.