PHYS 5040

Algebra and Topology in Physics

Spring 2021

Lecture 25

April 13

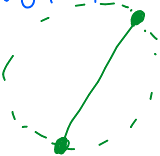
1 Review the various homotopy groups, and what kinds of topological defects they correspond to.

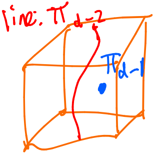
physical system in equilibrium characterized by $x \in X$

order parameter (x), O.P. space (X)

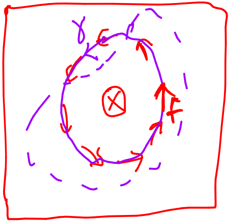
e.g. superfluid: $\psi \in \mathbb{C}$, with $|\psi| = \psi_0$: O.P. $\psi = \psi_0 e^{i\theta}$, $\theta \sim \theta + 2\pi$ $[X = S^1]$

nematic LC: $x \in \mathbb{RP}^2 = S^2/\mathbb{Z}_2$





point defect:

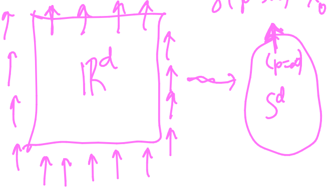


on all "loops", homotopy class of γ is same.

$f: \underbrace{\mathbb{R}^d - \{\text{point}\}}_{\text{def. ret. onto } S^{d-1}} \to X$

non-trivial point defects are classified by $\pi_{d-1}(X)$: homotopy classes of maps $\gamma: S^{d-1} \to X$.

"soliton" or "skyrmion" config...: $\gamma(p \to \infty) = x_0$, $p \in \mathbb{R}^d$



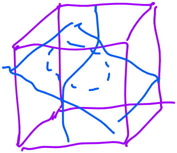
2 Draw three different kinds of topological defects in a nematic liquid crystal.

$[\pi_n(S^n) = \mathbb{Z}]$

Recall: nematic LC has O.P. space $\mathbb{RP}^2 = \underbrace{S^2}_{\text{simply connected}}/\mathbb{Z}_2$. ← free action

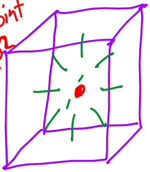
$\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$

line defect: (3d)



$\pi_2(\mathbb{RP}^2) = \pi_2(S^2) = \mathbb{Z}$

$\gamma : \mathbb{R}^3 - \text{point} \to \mathbb{RP}^2$



[each LC points radially outward]

"hedgehog"

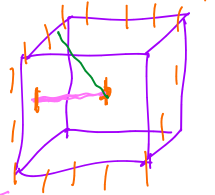
$\pi_3(\mathbb{RP}^2) = \pi_3(S^2) = \mathbb{Z}$

Hopf fibration

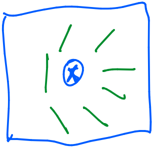
"$S^3 = S^2 \times S^1$"

S^1 ← locally

γ ← not homotopic to constant



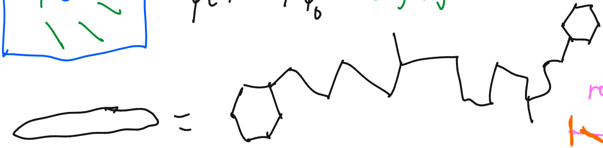
rotate



SF: polar coords.

$\psi = \psi(r) e^{i\theta}$

$\psi(0) = 0 \neq \psi_0$



3 Overview the goals of homology theory.

Thus far [algebraic top.]: $\boxed{\pi_1(X),}$ $\boxed{\pi_2(X), \ldots}$ . . .

Systematic computation

"hard" to calculate

if $\pi_n(X) \neq 0$ non-trivial defects surrounded by S^n.

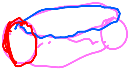
$\gamma : S^n \to X$ non-trivial

easier to compute? homology group $H_n(X)$

- if X is CW complex: $H_n(X)$ will depend only on n-cells, and gluing to/from $n \pm 1$ cells
- no n-cells $\Rightarrow H_n(X) = 0$.

Define: suppose $H_n(X) = \mathbb{Z} \times \cdots \times \mathbb{Z} \simeq \mathbb{Z}^k$. Then k different n-cycles $c_n^\alpha \subseteq X$, homeomorphic to S^n ... NOT retractible to a point.

e.g. $H_1(T^2) = \mathbb{Z}^2$



connection to cohomology...

for each c^α's, unique n-form $\omega^\alpha \sim \omega^\alpha + \underbrace{d\beta}_{\text{exact}}$

such that $\int_{c^\alpha} \omega^\alpha \neq 0$

4 Define a simplicial complex.

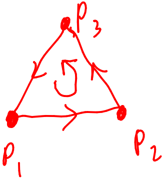
simplicial complex: CW complex built out of "higher dimensional triangles" (simplices)

0-simplex

Δ^0 = 

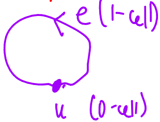
Δ^1 = 

$= (p_1, p_2)$

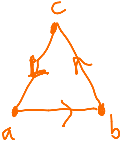
Δ^2 = 

$= (p_1 p_2 p_3)$

- rule #1: all boundary Δ^{n-1}'s in Δ^n must be distinct [no repeated points]
- rule #2: no 2 Δ^n's can have same boundary...

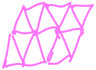
e.g. S^1: CW complex 

S^1 as simplicial complex:



[NOT filled in]

higher d:



5 Define the space of r-chains. What is the boundary operator ∂_r?

let $X = \{ \underbrace{\overset{c_0^\alpha}{p_0}, p_1, p_2, \dots}_{\Delta^0 s}, \underbrace{\overset{c_1^\alpha}{(p_0p_1)}, (p_1p_2), \dots}_{\Delta^1 s}, \dots \}$

simp. comp.

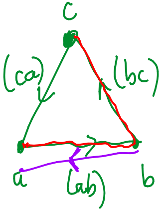
Formal: r-chain $\sigma = \sum n_\alpha c_r^\alpha$, $n_\alpha \in \mathbb{Z}$

orthogonal "vectors", one for each Δ^r

group of r-chains

$C_r(X) = \mathbb{Z} \times \cdots \times \mathbb{Z}$

one $\mathbb{Z}$ for each Δ^r.



$$\int_{(ab)\text{ then }(bc)} \overset{\text{1-form}}{\omega} = \int_{(ab)} \omega + \int_{(bc)} \omega$$

$\underbrace{(ab)\text{ then }(bc)}_{\text{1-chain } \sigma}$!

$\sigma = (ab) + (bc)$

formal inverse: $(ab) - (ab) = 0 \in C_r(X)$

Inverses: $(ba) = -(ab)$

$$\int_{(ab)} \omega + \int_{(ba)} \omega = \int_{c} \omega = 0$$

[Analogy: $\int_0^1 + \int_1^0 = \int_0^1 - \int_0^1 = 0$]

6 Show that $\partial_{r-1}\partial_r = 0$.

7 What if we replace simplicial complex with CW complex?