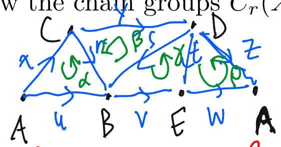
PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 26

April 15

1 Review the chain groups $C_r(X)$ on a simplicial complex.

$X =$ 

"like" $n_\alpha|\alpha\rangle + n_\beta|\beta\rangle + n_\gamma|\gamma\rangle + n_\rho|\rho\rangle$

(2-chains)

$C_2(X) = \mathbb{Z}^4$; $\sigma = n_\alpha \alpha + n_\beta \beta + n_\gamma \gamma + n_\rho \rho$, $n_\alpha \in \mathbb{Z}$

Motivation: if ω is a 2-form

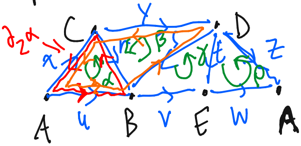
$$\int_\sigma \omega := n_\alpha \int_\alpha \omega + n_\beta \int_\beta \omega + n_\gamma \int_\gamma \omega + n_\rho \int_\rho \omega$$

$C_1(X) = \mathbb{Z}^9$: $\{u, v, w, x, r, s, t, y, z\}$

$C_0(X) = \mathbb{Z}^5$: $\{A, B, C, D, E\}$

2 What is the boundary operator ∂_r? (δ_r)

homomorphism: $\partial_r : C_r(X) \to C_{r-1}(X)$ $(r>0)$

$X =$ 

(2-dim simp. comp.)

$\partial_2 \alpha = u - r - x$

Motivation: ψ is a 1-form:

$\int_\alpha d_1\psi = \int_{\partial_2 \alpha} \psi$ Stoke's

$= \int_u \psi - \int_r \psi - \int_x \psi$

$\partial_2(\alpha+\beta) = u - s - y - x$

$= \partial_2\alpha + \partial_2\beta$

$= (u - r - x) + (-s - y + r)$

Denote $\alpha = (ABC)$; $u = (AB)$

$r = (CB)$; $x = (AC)$

$(AC) := -(CA)$

$\partial_2(ABC) = \underset{\text{[delete 1st]}}{(BC)} - \underset{\text{[delete 2nd]}}{(AC)} + \underset{\text{[del 3rd]}}{(AB)}$

$= (AB) + (BC) + (CA)$

$\partial_1(AB) = B - A$

3 Show that $\partial_{r-1}\partial_r = 0$.

delete p_j from sequence

$$\partial_r(p_0 \cdots p_r) = \sum_{j=0}^{r} (-1)^j (p_0 \cdots \hat{p}_j \cdots p_r)$$

Also: if swap two vertices

$$(p_0 p_1 \cdots p_r) = -(p_1 p_0 p_2 \cdots p_r)$$

Proof: chains add linearly... consider

$$\partial_{r-1}\partial_r (p_0 \cdots p_r)$$

$$= \partial_{r-1} \sum_{j=0}^{r} (-1)^j (p_0 \cdots \hat{p}_j \cdots p_r)$$

$$= \sum_{k<j} (-1)^k (-1)^j (p_0 \cdots \hat{p}_k \cdots \hat{p}_j \cdots p_r)$$

$$+ \sum_{k>j} (-1)^{k-1} (-1)^j (p_0 \cdots \hat{p}_j \cdots \hat{p}_k \cdots p_r)$$

$$= 0.$$

2 sums are equiv
[except for factor of -1 in #2]

Claim: $\sigma \in C_r(X)$.

$$\partial_{r-1}(\partial_r \sigma) = 0$$

Example:

$$\partial_2(p_0 p_1 p_2) = (p_1 p_2) - (p_0 p_2) + (p_0 p_1)$$

$$\partial_1(\ldots) = p_2 - p_1 - p_2 + p_0 + p_1 - p_0$$

$$= 0$$

4 Define the ~~homotopy~~ homology groups $\mathrm{H}_r(X)$ $\boxed{\text{exact}}$ sequence:

$$0 \xrightarrow{0} C_n(X) \to \cdots \to C_r(X) \xrightarrow{\partial_r} C_{r-1}(X) \xrightarrow{\partial_{r-1}} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{0} 0$$

if n-dim simp. complex

- $\mathrm{Im}(\partial_{n+1}) = 0$ $[H_n = \mathrm{Ker}(\partial_n)]$
- $\mathrm{Ker}(\partial_0) = C_0$ so $H_0(X) = C_0/\mathrm{Im}(\partial_1)$

Since $\partial_{r-1}\partial_r = 0$ [identity]

Subgroup $\mathrm{Im}(\partial_r) \leq C_{r-1}(X)$ (image) : $\sigma \in C_{r-1}$ $\exists\, \tilde{\sigma}$ such that $\partial_r \tilde{\sigma} = \sigma$

subgroup $\mathrm{Ker}(\partial_{r-1}) \leq C_{r-1}(X)$ (kernel) : $\sigma \in C_{r-1}$ such that $\partial_{r-1}\sigma = 0$.

$\mathrm{Im}(\partial_r) \leq \mathrm{Ker}(\partial_{r-1})$ $\quad \partial_{r-1}\underbrace{\partial_r \tilde{\sigma}}_{\mathrm{Im}} = 0 \; (\in \mathrm{Ker})$

All of groups are Abelian: normal subgroup

$$H_r(X) := \mathrm{Ker}(\partial_r)/\mathrm{Im}(\partial_{r+1}) \text{ is a group!}$$

r^{th} homology group

5 Calculate the non-vanishing homo~~topy~~logy groups for our example space X.

$X =$ 

$C_2 = \text{span}\{\alpha, \beta, \gamma, \rho\}$

$C_1 = \text{span}\{u, v, w, x, y, z, r, s, t\}$

$C_0 = \text{span}\{A, B, C, D, E\}$.

Since X is 2-dimensional simp. comp., so $H_2(X) = \text{Ker}(\partial_2)$

$\partial_2\alpha = u - r - x$

$\partial_2\beta = r - s - y$

$\partial_2\gamma = v - t + s$

$\partial_2\rho = w - z + t$

$\Rightarrow \text{Ker}(\partial_2) = 0$

$\Rightarrow H_2(X) = 0$

$\partial_2(\alpha + \beta) = (u - \cancel{r} - x) + (\cancel{r} - s - y)$

$\text{Im}(\partial_2) = \text{span}\{u - r - x,\ r - s - y,\ v - t + s,\ \underbrace{w - z + t}_{e_i}\}$

$\text{Ker}(\partial_1) = \text{span}\{\ \ ,\ \underbrace{u + v + w}_{f}\}$

$H_1(X) = \text{Ker}(\partial_1) / \text{Im}(\partial_2)$

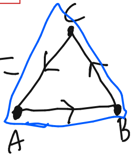
$= \{n\vec{f} + m_i\vec{e}_i\} / [\sigma \sim \sigma + m_i\vec{e}_i]$

$= \{n\vec{f}\} = \mathbb{Z}$

$H_0(X) = \mathbb{Z}$ [whenever is space is connected]

$\text{Im}(\partial_1) = \{B - A, C - A, D - A, E - A\}$

6 Calculate the non-vanishing homo~~topy~~logy groups for S^1.

$S^1 =$ 

Since 1-dimensional:

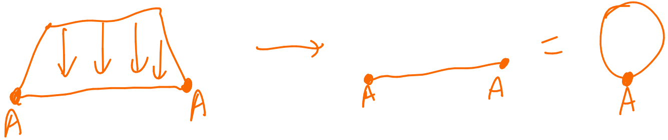
$$H_1(S^1) = \mathrm{Ker}(\partial_1)$$
$$= \{n((AB)+(BC)+(CA))\} = \mathbb{Z}$$

since connected: $H_0(S^1) = \mathbb{Z}$

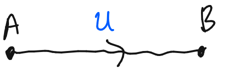
By convention: $H_n(S^1) = 0$ for $n > 1$.

Why S^1 and example X have all same H_n's!

- X can be deformation retracted to S^1

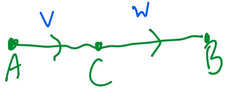


7 Argue that the homo~~topy~~logy groups do not depend on the specific simplicial complex (triangulation) used.



$C_1 = \mathbb{Z}$ (u)

$C_0 = \mathbb{Z}^2$ (A & B)



$C_1 = \mathbb{Z}^2$ (v & w)

$C_0 = \mathbb{Z}^3$ (A & B & C)

Before: $\mathrm{Im}(\partial_1) = \mathrm{span}(B-A)$

After: $\mathrm{span}(B-C, C-A) = \mathrm{span}(B-A, C-A)$

Before: $\mathrm{Ker}(\partial_0) = \mathrm{span}(A, B)$

After: $\mathrm{span}(A, B, C)$

H_0 before, $\{nA + mB\}/\{B \sim A\} = \{(n+m)A\} = \{nA\} = \mathbb{Z}$

after: $\{nA + mB + pC\}/[B \sim A, C \sim A] = \{(n+m+p)A\} = \mathbb{Z}$