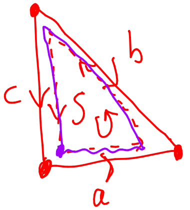
PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 27

April 20

1 Review the definition of homology groups $\mathrm{H}_n(X)$.

$$H_n(X) = \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1})$$



$$\partial_2 S = \boxed{a - b + c} \in \mathrm{Im}(\partial_2)$$

$$\int_S \underbrace{d\omega_1}_{2\text{-form}} = \int_{\partial_2 S} \omega_1 = \int_a \omega_1 - \int_b \omega_1 + \int_c \omega_1$$

$\mathrm{Ker}(\partial_n)$ = set of all closed n-dimenisonal surfaces [& sums]

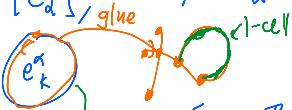
$\mathrm{Im}(\partial_{n+1})$ = n-dim surfaces which are themselves the bdy of (n+1)-dim region.

$\mathrm{Im}(\partial_{n+1}) \leq \mathrm{Ker}(\partial_n)$: spaces which are boundaries must have no bdy

2 Explain how to calculate the homology groups of a CW complex.

Why CW > simplicial? CW complex often has <u>few</u> <u>cells</u>

Suppose a CW complex w/ k-cells $\{e^k_\alpha\}$, identify ∂e^k_α / glue



"chain group": $C_k(X) = \underset{(e^k_\alpha)}{\mathbb{Z}} \times \underset{(e^k_\beta)}{\mathbb{Z}} \times \cdots$ [one $\mathbb{Z}$ for each cell]

$\partial_k : C_k(X) \to C_{k-1}(X)$: $\partial_k e^k_\alpha = \sum_{\substack{(k-1)\text{-cells} \\ e^{k-1}_\beta}} n_{\alpha,\beta}\, e^{k-1}_\beta$

gluing map $S^{k-1} \longmapsto S^{k-1}$ classified by $\pi_{k-1}(S^{k-1}) = \mathbb{Z}$.

$n_{\alpha,\beta}$ is an integer corresponds to homotopy class of gluing map $\partial e^k_\alpha \longmapsto e^{k-1}_\beta / \sim$

where $\sim$ identifies all points outside e^{k-1}_β together

$\boxed{3}$ Calculate the homology groups of S^n.

$n=1$:

$S^1 =$  e (1-cell), v (0-cell)

$\partial_1 g = v - u$

[compare w/ simp: ]

if $n>1$

$H_n(S^1) = 0$. no e^n_α cells in CW complex, $C_n(S^1)=0$,

$\partial_1 e = (\text{end}) - (\text{start})$
$= v - v = 0$

$H_1(S^1) = \mathrm{Ker}(\partial_1)/\underbrace{\mathrm{Im}(\partial_2)}_{=0 \text{ [nothing]}}$

$= \mathrm{Ker}(\partial_1) = \{ne, n \in \mathbb{Z}\} = \mathbb{Z}$

$H_0(S^1) = \underbrace{\mathrm{Ker}(\partial_0)}_{= C_0} / \underbrace{\mathrm{Im}(\partial_1)}_{0} = \mathbb{Z} = \{nv\}$ same

Ex: $n=2$;   f (n-cell), v (0-cell)

Case $n>1$:

$\partial f \xmapsto{\text{glue}} v$ $[\partial f \sim v]$

$C_0(S^n) = \underset{(v)}{\mathbb{Z}}$ $C_n(S^n) = \underset{(f)}{\mathbb{Z}}$

but $C_k(S^n) = 0$, if $k \neq 0, n$.

$\to H_k(S^n) = 0$ if $k \neq 0, n$.
[since $H_k \leq C_k$].

$H_n(S^n) = \mathrm{Ker}(\partial_n)/\overset{0}{\cancel{\mathrm{Im}(\partial_{n+1})}}$

because $C_{n-1} = 0$, and $\partial_n : C_n \to 0$

$\mathrm{Ker}(\partial_n) = C_n$

$H_n(S^n) = \mathbb{Z}$

$H_0(S^n) = \mathbb{Z}$

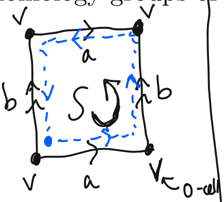
sneak peek:

$\int_{c_n} d\omega_{n-1} = \int_{\partial c_n} \omega_{n-1}$

exact n-form

4 Calculate the homology groups of the torus T^2, and the Klein bottle.

2d torus: T^2



$C_2 = \underset{[S]}{\mathbb{Z}}$

$C_1 = \underset{(a)}{\mathbb{Z}} \times \underset{(b)}{\mathbb{Z}}$

$C_0 = \underset{[v]}{\mathbb{Z}}$

$\partial_2 S = +a + b - a - b = 0$

$\partial_1 a = \underset{(end)}{v} - \underset{(start)}{v} = 0$

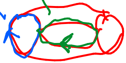
$\partial_1 b = 0$

no 3-cells

$H_2(T^2) = \mathrm{Ker}(\partial_2)/0 = C_2 = \mathbb{Z}$

$H_1(T^2) = \mathrm{Ker}(\partial_1)/\langle \partial_2 S\rangle$ $= \mathrm{Im}(\partial_2)$

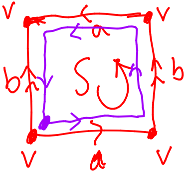
$= C_1/0 = \mathbb{Z} \times \mathbb{Z}$



$H_0(T^2) = C_0/0 = \mathbb{Z}$

$b_0 = 1$ $b_1 = 2$ $b_2 = 1$

Klein bottle:



$C_2 = \mathbb{Z}$

$C_1 = \mathbb{Z}^2$

$C_0 = \mathbb{Z}$

$\partial_2 S = a + b + a - b = 2a$

$\partial_1 a = \partial_1 b = v - v = 0$

$H_2(K) = \mathrm{Ker}(\partial_2) = 0$

$H_1(K) = C_1 / \mathrm{Im}(\partial_2)$

$= \{na + mb\}/\{2n'a\}$

not orientable

$n \sim n + 2n' \Rightarrow n \in \mathbb{Z}_2$

$= \underset{(a)}{\mathbb{Z}_2} \times \underset{(b)}{\mathbb{Z}}$

$H_0(K) = C_0/0 = \mathbb{Z}$

5 Find the homology groups of the two dimensional Riemann surface of genus g.

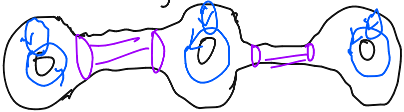
Σ_g

non-trivial generator of H_1?

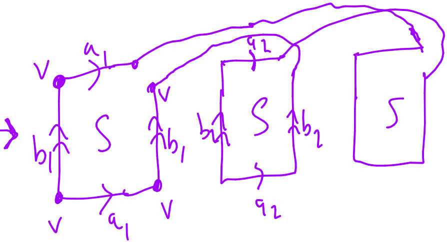
$H_1(\Sigma_g) = \mathbb{Z}^{2g}$ ✓

$H_0(\Sigma_g) = \mathbb{Z}$ ✓ [connected]

$H_2(\Sigma_g) = \mathbb{Z}$ ✓



$[g=0 : S^2 ; \; g=1 : T^2]$



$\partial_1 a_j = 0 = v - v$

$\partial_1 b_j = 0$

$\partial_2 S = a_1 + b_1 - a_1 - b_1 + a_2 + b_2 - a_2 - b_2 + \cdots = 0$

6 Relate the Euler characteristic to homology groups.

Define: if simplicial complex X has m_n Δ^n's in it:

Euler characteristic: $\chi = m_0 - m_1 + m_2 - m_3 + \cdots$

[2d]: $\chi = V - E + F$

Claim: Suppose $H_n(X) = \mathbb{Z}^{b_n}$ $\quad b_n = n^{th}$ Betti number

Then: $\chi = b_0 - b_1 + b_2 - b_3 + \cdots$

Proof: $C_n(X) = \mathbb{Z}^{m_n}$ $\quad \partial_n : \underbrace{\mathbb{Z}^{m_n}}_{C_n} \to \underbrace{\mathbb{Z}^{m_{n-1}}}_{C_{n-1}}$ "matrix"

rank-nullity: $\mathrm{Ker}(\partial_n) = \mathbb{Z}^{r_n}$ and $\mathrm{Im}(\partial_n) = \mathbb{Z}^{s_n} \Rightarrow \underline{r_n + s_n = m_n}$

$H_n(X) = \mathrm{Ker}(\partial_n) / \mathrm{Im}(\partial_{n+1}) \Rightarrow b_n = r_n - s_{n+1}$

$\chi = m_0 - m_1 + m_2 - \cdots = \underset{\substack{\| \\ m_0}}{r_0} - (s_1 + r_1) + (s_2 + r_2) - \cdots$

$= (r_0 - s_1) - (r_1 - s_2) + (r_2 - s_3) - \cdots = b_0 - b_1 + b_2 - \cdots$