PHYS 5040
Algebra and Topology in Physics
Spring 2021

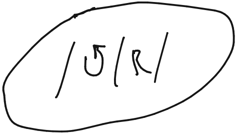
Lecture 28

April 22

1 Review Stokes' Theorem in the language of chains and differential forms. Review closed and exact forms.

multivar calculus: Stokes' Thm: $\int_R (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_{\partial R} \vec{v} \cdot d\vec{s}$

2d $R \subseteq$ 3d space



in the language of differential forms:

$$\int_R \underbrace{dV}_{\text{2-form}} = \int_{\partial R} V$$

d: exterior derivative

V: 1-form

R: 2-dim orient surface

∂R: 1-dim oriented surface

generalize to all dimensions & p-forms for any p

[R will be a (p+1)-dim surface]

∂ corresponds to "same" bdy operation as in homology

exact differential form: $\omega_p = d_{p-1}\alpha_{p-1}$ (ω_p: p-form; α_{p-1}: (p-1) form to p-form)

closed differential form: $d_p \omega_p = 0$

$d_{p+1} d_p \omega_p = 0$ $[d^2 = 0]$

set of exact diff. forms: $B_p(X)$ [X is manifold]

set of closed diff. forms: $Z_p(X)$

so $B_p(X) \subseteq Z_p(X)$

2 Write down an exact sequence involving differential forms and exterior derivatives, and define the (de Rham) cohomology groups $\mathrm{H}^n(X)$.

$Z_p(X)$ (closed) & $B_p(X)$ (exact) are Abelian groups; binary operation is addition

e.g. if $\omega_{1,2} \in Z_p(X)$, then $d_p(\omega_1 + \omega_2) = d_p\omega_1 + d_p\omega_2 = 0 + 0 = 0$

[identity element is 0] ; $B_p(X) \leq Z_p(X)$

let's define $F_p(X) =$ [Abelian] group of p-forms under additions [$B_p, Z_p \leq F_p$]

$$\cdots \longrightarrow F_{p-1} \xrightarrow{d_{p-1}} F_p \xrightarrow{d_p} F_{p+1} \longrightarrow \cdots$$

d_p is a homomorphism: $d_p(\underset{\in F_p}{\omega_1} + \underset{\in F_p}{\omega_2}) = d_p\omega_1 + d_p\omega_2$

AND: $d_{p+1} d_p = 0 \implies$ exact sequence [just like homology]

- $\mathrm{Im}(d_p) = B_{p+1}(X)$ [$\psi_{p+1} = d_p\omega_p$]
- $\mathrm{Ker}(d_p) = Z_p(X)$ [$d_p\omega_p = 0$]

(de Rham) cohomology groups:

$$H^p(X) := Z_p(X) / B_{p-1}(X)$$

3 Show that if X is connected, $\mathrm{H}^0(X) = \mathbb{R}$.

elements of $H^p(X)$ are $[\omega]$, equivalence classes of "cohomologous" forms:

$$\omega_p \sim \omega_p + d_{p-1}\lambda_{p-1}$$

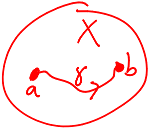
Suppose: $d_0 f = 0$. In local coordinates x_i ...

scalar

$$d_0 f = \sum_i \frac{\partial f}{\partial x_i} dx_i = 0 \iff \frac{\partial f}{\partial x_i} = 0 \iff f = \text{const.}$$

$\Rightarrow$ space of cohomologous closed 0-forms = constant functions

Proof #2: let γ be a line segment in X.



Stokes' Thm: $\int_\gamma d_0 f = \int_{\partial\gamma} f = f(b) - f(a) = 0$ if $d_0 f = 0$

$f(b) = f(a)$ if a path connects a & b.

constant

$f(x_i) = c$ if f is closed 0-form, $H^0(X) = \{\text{group of closed 0-forms}\} / 0$

trivial group: no (-1)-form

$= \mathbb{R}$

4 Argue that if $\mathrm{H}_n(X) = \mathbb{Z}^k$, then $\mathrm{H}^n(X) = \mathbb{R}^k$.

[no $\times \mathbb{Z}_2$]

Poincaré duality: manifold X, then: 1) $H_n(X) = \mathbb{Z}^k$ (hom) for some k

2) $H^n(X) = \mathbb{R}^k$ (cohom), same k

Let $\gamma_1, \ldots, \gamma_k$ denote n-chains that are not bdy of $(n+1)$-chain.

$\gamma_1 \sim \gamma_1 + \partial_{n+1} R$ (R: $(n+1)$-chain)

"Inner product" $H_n \times H^n \to \mathbb{R}$: $\langle \gamma | \omega \rangle = \int_\gamma \omega$.

Note: $\int_{\gamma_p} (\omega_p + d_{p-1}\lambda_{p-1}) = \int_{\gamma_p} \omega_p + \int_{\partial_p \gamma_p = 0} \lambda_{p-1} = \int_{\gamma_p} \omega_p$

$\int_{\gamma_p + \partial_{p+1} R_{p+1}} \omega_p = \int_{\gamma_p} \omega_p + \int_{R_{p+1}} \underbrace{d_p \omega_p}_{=0} = \int_{\gamma_p} \omega_p$

Claim: "inner product" is non-degenerate:

if $\langle \tilde{\gamma} | \omega \rangle = 0 \ \forall \omega$, then $\tilde{\gamma} = 0$.

vice versa: $\langle \gamma | \tilde{\omega} \rangle = 0 \ \forall \gamma$, then $\tilde{\omega} = 0$.

5 For $n = 1, 2$, find the non-trivial element of $\mathrm{H}^n(\mathrm{S}^n)$.

$H^0(S^1) = H^1(S^1) = \mathbb{R}$

[NB: $H^n(S^1) = 0$ if $n \geq 2$ since $d\theta \wedge d\theta = 0 \ldots$]

$H^0(S^1) = \{ f(x) = c \}$ const.

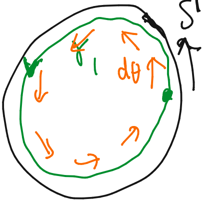
$H^1(S^1) = \mathbb{R}$: look for 1-form ω_1 s.t. if γ_1 is non-contractible loop in S^1,

$$\int_{\gamma_1} \omega_1 \neq 0 .$$

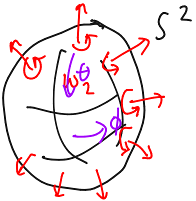
$$\int_0^{2\pi} d\theta = 2\pi$$

$d\theta \in H^1(S^1)$

~ "length" of non-trivial closed chain



$\underbrace{H^0(S^2)}_{\text{const.}} = \underbrace{H^2(S^2)}_{???} = \mathbb{R}$



$\int_{S^2} \omega_2 = \mathrm{Vol}(S^2)$ must exist.

(in 2d, area...)

$\omega_2 = \sin\theta \, d\theta \wedge d\phi$

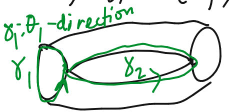
$$\int_{S^2} \omega_2 = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta = 4\pi$$

$\omega_2 \neq d(\underbrace{-\cos\theta \, d\phi}_{\text{NOT defined at } \theta = 0, \pi})$

6 Find all non-trivial closed but not exact forms on T^2.

"donut" or T^2 : $(\theta_1, \theta_2) \sim (\theta_1 + 2\pi, \theta_2) \sim (\theta_1, \theta_2 + 2\pi)$

$= S^1 \times S^1$



elements of $H^0(T^2) = \mathbb{R}$: 1

elements of $H^1(T^2) = \mathbb{R}^2$: $c_1 d\theta_1 + c_2 d\theta_2$

elements of $H^2(T^2) = \mathbb{R}$: $d\theta_1 \wedge d\theta_2$ [volume form] :

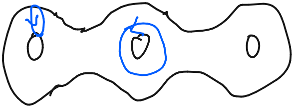
if $\gamma_1(t) = (2\pi t, 0)$ $\quad 0 \le t \le 1$

$$\int_{\gamma_1} (c_1 d\theta_1 + c_2 d\theta_2) = \int_0^1 \left(c_1 \frac{d\theta_1}{dt} + c_2 \frac{d\theta_2}{dt}\right) dt = 2\pi c_1$$

$$\int_{T^2} d\theta_1 \wedge d\theta_2 = (2\pi)^2$$

7 Consider electromagnetism on the Riemann surface of genus g. Focusing on the vector potential A, determine the number of "pure gauge" modes that don't contribute to the magnetic field.

e.g. $g=3$

$\Sigma_g =$ 

$H_1(\Sigma_g) = \mathbb{Z}^{2g}$

$\Rightarrow H^1(\Sigma_g) = \mathbb{R}^{2g}$

Physical DOF in magnetic field $B = dA$ [not including timelike component in A]

in magnetostatics, B physical, A is not...

$$\underbrace{A}_{\text{1-form}} = \underbrace{d\phi}_{\text{exact}} + \underbrace{A_{phys}}_{dA_{phys} \neq 0} + \underbrace{A_{top}}_{\text{closed but not exact forms from } H^1(\Sigma_g)}$$

numerics: A_{top} are <u>further</u> redundancy than naive expectation