PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 29

April 27

1 Review the cohomology groups $\mathrm{H}^k(X)$. How do we interpret them in terms of differential forms?

$$H^k(X) = \left\{ \text{closed } k\text{-forms } \omega_k \text{ on } X \overset{\text{such that}}{:} d_k \omega_k = 0 \right\} \Big/ \left[\omega_k \sim \omega_k + \overbrace{d_{k-1}\alpha_{k-1}}^{\text{exact}} \right]$$

(de Rham)

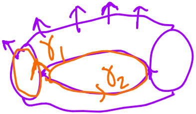
for manifold X: $H^k(X) = \mathbb{R}^{b_k}$ $\quad b_k = k^{th}$ Betti #, because

$[\omega] \in H^k(X)$ of the form $\omega = \sum_{j=1}^{b_k} c_j \alpha_j$

$c_j \in \mathbb{R}$ $\quad \alpha_j$: "topological" forms

Recall: homology $H_k(X) = \mathbb{Z}^{b_k}$ $\quad$ same Betti number

T^2: $H^2(T^2) = \mathbb{R} \leftarrow \int_{T^2} \omega_2 \neq 0$



$H^1(T^2) = \mathbb{R}^2$

$\gamma_1 : \alpha_1 = d\theta_1 \qquad [\theta_1 \sim \theta_1 + 2\pi]$

$\gamma_2 : \alpha_2 = d\theta_2$

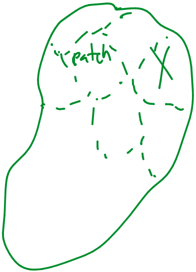
$$\int_{\gamma_i} \alpha_j = 2\pi \delta_{ij}$$

2 Let X be a closed n-dimensional manifold. Why does $\mathrm{H}^0(X) = \mathrm{H}^n(X) = \mathbb{R}$?

Recall: (connected) manifold X . . . $H^0(X) = \mathbb{R}$:
only closed 0-forms, scalar functions f s.t. $df = 0$
are $f(x) = \text{const.}$

Claim: $H^n(X) = \mathbb{R}$. [X is compact]
• a single [up to exact] "non-trivial" n-form on X.

$$\text{Volume}(X) = \sum \underbrace{\text{patch volume}}_{>0} = \int_X \overset{n\text{-form}}{\omega_{vol}} \neq 0$$



Could $\omega_{vol} = d_{n-1}\alpha_{n-1}$?

If so: $\displaystyle\int_X d_{n-1}\alpha_{n-1} = \int_{\partial_n X} \alpha_{n-1} = 0$ b/c $\partial_n X = 0$ (closed/compact.)

NO! $[\omega_{vol}] \in H^n(X)$. (non-identity)

If there were 2 generators of $H^n(X)$ . . . $\displaystyle\int_{X'} \omega_{vol} = 0$? $\mathrm{Vol}(X') > 0$???

3 Explain Poincaré duality: $\underline{H^k(X) = H^{n-k}(X).}$ $[b_k = b_{n-k}]$

X: closed & compact

If $[\alpha] \in H^k(X)$, $[\beta] \in H^{n-k}(X)$. Define $\langle \alpha | \beta \rangle = \int_X \alpha \wedge \beta$.

Well-defined: $\int_X (\alpha + d\lambda) \wedge \beta = \int_X \alpha \wedge \beta + \underbrace{\cancel{\int_X d\lambda \wedge \beta}}_{} \to 0$

$= \int_X d(\lambda \wedge \beta) - (-1)^k \lambda \wedge \cancel{d\beta} \to 0$

$\hookrightarrow \int_{\partial X} \lambda \wedge \beta = 0$ as $\partial X = 0$

Argue: if $[\beta] \neq [0]$, $\exists$ α s.t. $\int_X \alpha \wedge \beta \neq 0$:

$\int_X ?? \wedge \beta = \int_X \omega_{vol}$

$\omega_{vol} = \rho(\vec{x}) dx_1 \wedge \cdots \wedge dx_n$

$\beta = \cdots$

$+ \chi(x)\, dx_{k+1} \wedge \cdots \wedge dx_n$

Do this argument for α to o....

"vector spaces" H^k, H^{n-k} have same dimension.

$\alpha = \frac{\rho}{\chi} dx_1 \wedge \cdots \wedge dx_k$;

$\alpha \wedge \beta = \frac{\rho}{\cancel{\chi}} dx_1 \wedge \cdots \wedge dx_k \wedge \cancel{\chi}\, dx_{k+1} \cdots \wedge dx_n = \omega_{vol}$ ✓

picks out terms w/ no $dx_1 \ldots dx_k$

$\boxed{4}$ What is a (mathematical) ring?

ring R = group w/ TWO binary operations:

addition (Abelian) & multiplication

$\hookrightarrow$ $a, b \in R$: $a + b \in R$

inverse of a: $-a$ [additive identity $= 0$]

multiplication does NOT need to give a group.

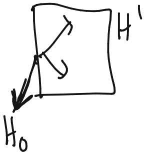
DO require: $c(a+b) = ca + cb$ [distributive]

examples of rings: $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{Z}_n$, ...

5 Why do the cohomology groups form a ring?

cohomology ring: $H^*(X) = \{$ formal sums of elements of $H^0, H^1, \dots, H^n \}$

$$= H^0(X) \oplus H^1(X) \oplus \cdots \oplus H^n(X)$$



addition: $\omega_1 = 1 + \cdots$
$\omega_2 = -\frac{1}{2} + \cdots$

$$= \left(1 - \frac{1}{2}\right) + \cdots = 1/2 \quad \checkmark$$

multiplication: wedge product!

$$\underbrace{dx_1}_{H^1} \wedge \underbrace{(dx_3 \wedge dx_4)}_{H^2} = \underbrace{dx_1 \wedge dx_3 \wedge dx_4}_{H^3}$$

but $dx_1 \wedge dx_1 = 0 \in H^*$ [additive identity]

6 Calculate the homology groups of the n-dimensional torus T^n.

1) co-homology instead! $H^k(T^n) = ?$ $\quad T^n = \underbrace{S^1 \times S^1 \times \dots \times S^1}_{n \text{ times}}$

2) go for $H^*(T^n)$.

"suggestive" but precise notation.

3) <u>Lemma:</u> $H^*(X \times Y) = H^*(X) \wedge H^*(Y)$

$\alpha \in H^*(X)$, $\beta \in H^*(Y)$: $\quad \alpha \wedge \beta \in H^*(X \times Y)$

$(x_1, \dots, x_m)$ on X , $(y_1, \dots, y_n)$ on Y

$(x_1, \dots, x_m, y_1, \dots, y_n)$ on $X \times Y$

4) $H^*(T^n) = \underbrace{H^*(S^1) \wedge \dots \wedge H^*(S^1)}_{n \text{ times}}$

$H^*(S^1) = a_1 + b_1 d\theta_1$, $\quad a_n + b_n d\theta_n$

in $H^*(T^n)$... k-forms of the form $d\theta_{i_1} \wedge \dots \wedge d\theta_{i_k}$ ... $\binom{n}{k}$ of them

5) $H^k(T^n) = \mathbb{R}^{\binom{n}{k}} \Rightarrow H_k(T^n) = \mathbb{Z}^{\binom{n}{k}}$

7 Calculate the homology groups of $S^2 \times S^1$.

use cohomology ring:

$$H^*(S^2 \times S^1) = \underbrace{H^*(S^2)}_{\alpha + \beta \omega_2} \wedge \underbrace{H^*(S^1)}_{a + b\,d\theta}$$

"volume" form

$$H^*(S^2 \times S^1) = \underset{(1 \wedge 1)}{\overset{(H^0)}{A \cdot 1}} + \underset{(1 \wedge d\theta)}{\overset{(H^1)}{B\,d\theta}} + \underset{(\omega_2 \wedge 1)}{\overset{(H^2)}{C\,\omega_2}} + \overset{(H^3)}{D\,\omega_2 \wedge d\theta}$$

<u>cohomology</u>

$H^0(S^2 \times S^1) = \mathbb{R}$

$H^1(S^2 \times S^1) = \mathbb{R}$

$H^2(S^2 \times S^1) = \mathbb{R}$

$H^3(S^2 \times S^1) = \mathbb{R}$

$\Longrightarrow$

homology

$H_0 = \mathbb{Z}$

$H_1 = \mathbb{Z}$

$H_2 = \mathbb{Z}$

$H_3 = \mathbb{Z}$

8 Through which of the following manifolds can we thread magnetic flux across the entire manifold: S^2, T^3, $\mathbb{C}P^2$, S^4?

Looking for manifolds has $H^2(X) \neq 0$.

If so, set $B = dA \sim \omega_2 \in H^2(X)$

$$\Phi_B = \int_{\gamma_2 \subseteq X} B \neq 0$$

$H^2(S^2) = \mathbb{R}$ ✓

$H^2(T^3) = \mathbb{R}^3$ ✓

$[d\theta_1 \wedge d\theta_2,\ d\theta_1 \wedge d\theta_3,\ d\theta_2 \wedge d\theta_3]$

$H^2(\mathbb{C}P^2) = \mathbb{R}$ ✓

$H^2(S^4) = 0$ ✗