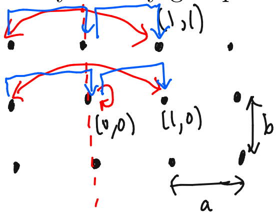
# PHYS 5040
# Algebra and Topology in Physics
# Spring 2021

# Lecture 3

January 21

**1** Consider an ionic lattice in two dimensions consisting of identical atoms arranged at the points $na\hat{\mathbf{x}} + mb\hat{\mathbf{y}}$ for $n, m \in \mathbb{Z}$. What is the symmetry group of the lattice?




reflections: $r_x \cdot (n,m) = (-n, m)$

$r_y \cdot (n,m) = (n, -m)$

translation: $t_{n',m'} \cdot (n,m) = (n+n', m+m')$

& any combinations of the above!

**2** What is the group of transformations that leave $(0,0)$ invariant?

unique 4 pts dist. $\sqrt{a^2+b^2}$


Case 1 $(a \neq b)$

- map 4 pts to each other
- map vertical edge $\to$ vert
  horizontal edge $\to$ hor
- symmetry group $D_4$ (Zee $D_2$)

$D_4 = \{1, r, s, sr\}$

$r^2 = 1, \; s^2 = 1, \; sr = rs$

1) map lattice to itself [full symmetry G]

AND

2) map $(0,0)$ to itself

Case 2 $(a = b)$

- $90°$ rotation OK
- symmetry group $\to D_8$

$D_8 = \langle \underbrace{r, s}_{\text{generated by}} \mid \underbrace{r^4 = 1, s^2 = 1, rs = sr^3}_{\text{"constraints"}} \rangle$

"constraints": $r^4 = 1$

group presentation

**3** Define a subgroup and a stabilizer group.

subgroup: $G$ is group. $H \subseteq G$, and $H$ is subgroup if

$\subseteq$ is contained in

1) $g_1, g_2 \in H \Rightarrow g_1 g_2 \in H$

2) associative

3) $1 \in H$

4) $g \in H \iff g^{-1} \in H$

$\iff$ if and only if

$H$ is a subgroup of $H$: $H \leq G$

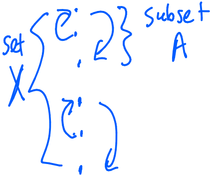
$D_4 \leq D_8$

$D_8 = \{1, r, r^2, r^3, s, sr \ldots\}$

then $D_4 = \{1, r^2, s, sr^2\}$

$r = 90^\circ$ rot. $r^2 = 180^\circ$ rot.

stabilizer / little group: if $G$ acts on $X$, stabilizer of $A \subseteq X$




is $H = \{g \in G : g \cdot a_1 = a_2\}$

":" such that; $a_1$: A; $a_2$: A

point group [of lattice] = stabilizer of $(0,0)$

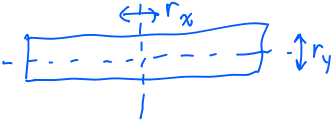
**4** Define a product group. Explain why $D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

product group: $G \times H = \{(g, h) : g \in G, h \in H\}$

groups

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

$D_4$ = symmetry group of rectangle:




$D_4 = \{1, r_x, r_y, r_x r_y\}$

$r_x \times r_y = r_x r_y$

$r_y \times r_x = r_x r_y$

$\underbrace{r_y \times r_x r_y = r_x}$

$(1, r_y)(r_x, r_y) = (r_x, 1)$

$= \{(1,1), (r_x, 1), (1, r_y), (r_x, r_y)\}$

$D_4 = \{1, r_x\} \times \{1, r_y\}$

$r_x^2 = 1 \qquad r_y^2 = 1$

$= \mathbb{Z}_2 \times \mathbb{Z}_2$

$1 \times 1 = 1$

$1 \times r_x = r_x$

$r_x \times 1 = r_x$

$r_x^2 = 1$

$\boxed{5}$ Is the symmetry group of the lattice a product group?

$(a \neq b)$

$G \neq \{\text{translations}\} \times \{\text{reflections}\}$

$\in \mathbb{Z}$

$t_{n',m'} \cdot (n, m) = (n+n', m+m')$

$r_x \cdot (n,m) = (-n, m)$

$r_y \cdot (n,m) = (n, -m)$

$r_x t_{n',m'} \cdot (n,m) = (-n-n', m+m')$

$t_{n',m'} r_x \cdot (n,m) = (-n+n', m+m')$

$r_x t_{n',m'} = t_{-n',m'} r_x$

$G$ is non-Abelian

$\{\text{translations}\} = \mathbb{Z} \times \mathbb{Z} \simeq \mathbb{Z}^2$

$\{\text{reflections}\} = D_4 = \mathbb{Z}_2^2$

$t_{n_1,m_1} t_{n_2,m_2} = t_{n_1+n_2, m_1+m_2}$

Abelian groups

if $K$ & $H$ Abelian... $K \times H$ is Abelian:

$(k_1, h_1)(k_2, h_2) = (k_1 k_2, h_1 h_2) = (k_2 k_1, h_2 h_1) = (k_2, h_2)(k_1, h_1)$

**6** Define the semidirect product.

$$t_{n',m'} \cdot (n,m) = (n+n', m+m')$$

$$r_x \cdot (n,m) = (-n, m)$$
$$r_y \cdot (n,m) = (n, -m)$$

Why not a product group? $(t_{n'm'}, r)$, $r \in \{1, r_x, r_y, r_x r_y\}$

means reflect then translate

$$(t_{n',m'}, r_x) \cdot (n,m) = (n'-n, m'+m)$$

$t_{0,0}$

$$(1, r_x)(t_{n',m'}, 1) = (t_{-n',m'}, 1)(1, r_x) = (t_{-n',m'}, r_x)$$

$$(t_1, r_1)(t_2, r_2) := (t_1[r_1 \cdot t_2], r_1 r_2)$$

define as $(\equiv)$

$$r_x \cdot t_{n,m} = t_{-n,m}$$
$$r_y \cdot t_{n,m} = t_{n,-m}$$

groups

Semidirect product: given $H$ and $K$ AND group action of $K$ on $H$,

then $G = H \rtimes K$ is a group: $(h_1, k_1)(h_2, k_2) = (h_1[k_1 \cdot h_2], k_1 k_2)$

**7** What is the full symmetry group of the lattice?

$$G = \{\text{translations}\} \rtimes \{\text{reflections}\}$$
$$= \mathbb{Z}^2 \rtimes D_4$$

spatial (not time)

1) for "most" lattice in any dimension $d$

$$G = \mathbb{Z}^d \rtimes \text{point group}$$

2) point groups come from a finite list

[crystallographic restriction thm]

addition

$\mathbb{R} \neq \mathbb{R}^\times$

identity

$(-1)^2 = 1$

$x^2 = 1$ has 2 sol'ns in $\mathbb{R}^\times$

$x + x = 0$ $x = 0$ only