PHYS 5040
Algebra and Topology in Physics
Spring 2021

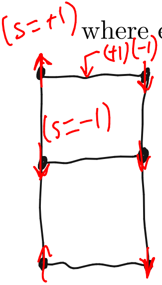
Lecture 4

January 26

$\boxed{1}$ Consider the following Ising Hamiltonian on a generic lattice:

$$H = -J \sum_{u,v \text{ neighbors}} s_u s_v$$

where each $s_v \in \{\pm 1\}$. What is the symmetry group G of H, and why?



$(E = -J(-1) = +J)$

What's symmetry group G of Ham. H?

- G to act on configurations $\underbrace{(s_1, \ldots, s_6)}_{\vec{s}}$

$$H(g \cdot \vec{s}) = H(\vec{s}) \checkmark$$

- implicit: identify group G <u>before</u> finding all energy states...

$\vec{s} \longrightarrow -\vec{s}$: then $s_u s_v \to (-s_u)(-s_v) = s_u s_v$ ✓

since $(-1)^2 = +1$ (identity)
spin flip generates 2-element group: $\mathbb{Z}_2$ ← $\{+1, -1\}$, group op is multiplications

2 Suppose $\boxed{J > 0.}$ What are the ground states of H? What happens to the symmetry?

$$H = -J \sum_{\substack{u,v \\ \text{neigh}}} s_u s_v \qquad s_u, s_v \in \{\pm 1\}$$

ground states of H: $\vec{s}_+ = (+1, +1, \dots)$

$\vec{s}_- = (-1, -1, \dots)$

(classical) system pick one.

$(-1)\cdot \vec{s}_+ = -\vec{s}_+ = \vec{s}_-$ AND $(-1)\cdot \vec{s}_- = \vec{s}_+$

$\mathbb{Z}_2$ symmetry is spontaneous broken: state has less symmetry than the EOM"

H symmetry preserving:

$E_1 < E_2 < E_3 < \dots$

$g \in G$

H symmetry breaking:

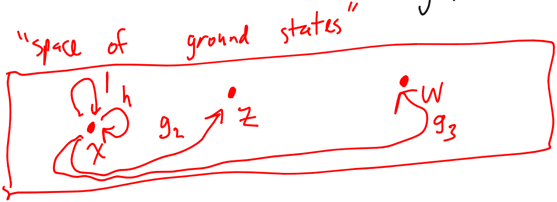
$E_1 < E_2 < E_3 < \dots$

$g \quad g$

3 Define a left/right coset.

Hamiltonian w/ symmetry group G $\xrightarrow{\text{ground state } x^{(\vec{s})}}$ invariant under subgroup $H \leq G$

$g \cdot x = x$ if and only if $g \in H$.



ground states	group elements
x	$H = \{1, h, \ldots\}$
z	$g_2 H = \{g_2, \ldots\}$
w	$g_3 H = \{g_3, \ldots\}$

Apply $g_2 h \cdot x$

$$= g_2 \cdot (h \cdot x)$$

$$= g_2 \cdot x = z$$

Since $g_2 h$ always $x \to z$, define (right) coset

$$g_2 H = \{g_2 h : h \in H\}$$

4 Show that cosets partition a group, and all have the same size.

1) Cosets partition group: $G = H \cup g_2H \cup g_3H$

AND $g_2H \cap H_2 = \text{empty} = \emptyset$

Claim: $\forall g \in G$, write coset gH [might be new]

Suppose $g_2 \notin H$. Look at g_2H.

Suppose that $g_2'H \cap g_2H \neq \emptyset$

$g_2'h_1 = g_2h_2$

$g_2' = g_2\underbrace{(h_2h_1^{-1})}_{\in H} \in g_2H$

$g_2'H = g_2\underbrace{(h_2h_1^{-1}H)}_{=H}$

2) [for finite groups] cosets have same # of elements: ✓

Proof: If $H = G$... one coset

let's assume $g \notin H$.

$gH \neq H$ # of elements of H

• at most $|H|$ elements in gH:

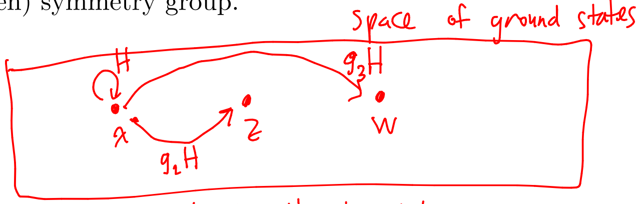
$gH = \{g1, gh_1, gh_2, \ldots\}$

• suppose $gh_1 = gh_2$, then $\cancel{g^{-1}}gh_1 = \cancel{g^{-1}}gh_2$ $(h_1 = h_2)$

Corollary (Lagrange's Thm): $|G|$ is divided by $|H|$

# of elements

5 Explain spontaneous symmetry breaking, and why the ground states of any Hamiltonian are in one-to-one correspondence with cosets of its (broken) symmetry group.



$|H| = |g_2H| = |g_3H|$

degenerate ground states related by full symmetry group G are one-to-one w/ cosets of the unbroken subgroup

The set of (right) cosets is G/H

in general, <u>not</u> a group

One can also define (left) cosets Hg

Set of these $H \backslash G$

6 When do the cosets form a group?

We'd need: $(g_1 H)(g_2 H) = g_3 H$

$$\boxed{(g_1 H)(g_2 H) = g_1 g_2 H}$$

$g_1 g_2 = g_3$

$(g_1 h_1)(g_2 h_2) = g_3 h_3 = g_1 g_2 h_3$

$h_1 g_2 h_2 = g_2 h_3$

$g_2^{-1} h_1 g_2 h_2 = h_3$

$\rightarrow g_2^{-1} h_1 \underbrace{g_2}_{\text{arbitrary} \in G} = h_3 h_2^{-1} \in H$

or

conjugation by g_2

$g_2^{-1} H g_2 = H$

$\forall g_2 \in G$

- two elements of a group are conjugate if $g_1 = g^{-1} g_2 \underbrace{g}_{\text{arbitrary}}$
  $g_1 \sim g_2$
- the sets of group elements that are conjugate form conjugacy classes

7 Define a normal subgroup.

We say if $g^{-1}Hg = H \quad \forall g \in G$, H is a <u>normal</u> subgroup of G;

$$H \trianglelefteq G$$

quotient group: G/H is a subgroup of G

another group

Theorem: If $\phi : G \to K$ is homomorphism, then
$\ker(\phi) := \{ g \in G : \phi(g) = 1 \}$ obeys $\ker(\phi) \trianglelefteq G$.

Proof: $\ker(\phi)$ is a subgroup: $\phi(\underbrace{g_1, g_2}_{\in \ker}) = \phi(g_1)\phi(g_2) = 1 \cdot 1 = 1$ ✓

$\phi(g^{-1} \underset{\in \ker}{h} g) = \phi(g^{-1})\phi(h)\phi(g) = \phi(g)^{-1} 1 \; \phi(g)$
$= \phi(g)^{-1}\phi(g) = 1$

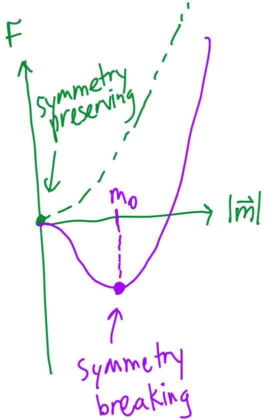
By definition, $g^{-1}hg \in \ker(\phi)$

8 Consider a 3-dimensional isotropic magnet. Write down the free energy and describe the possible patterns of symmetry breaking.

order parameter: magnetization $\vec{m} = (m_x, m_y, m_z)$

look for minimum $F(\vec{m})$

rotation invariance of universe + reflections: $O(3)$ $\rightarrow$ $F(|\vec{m}|)$



minima of F are classified by vectors of length $|\vec{m}| = m_0$

unbroken symmetry: $O(2)$

$O(3)/O(2) = S^2$ (2-sphere)